微積分 MA1001-A 上課筆記（精簡版）

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Ching-hsiao Arthur Cheng 鄭經斅

Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuous function, and R be the region enclosed by the graph of the function f, the x-axis and straight lines $x = a$ and $x = b$. We are interested in the area of R which is denoted by $\mathcal{A}(\mathrm{R})$.

Partition $[a, b]$ into n sub-intervals with equal length, and let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$. By the Extreme Value Theorem, for each $1 \leqslant i \leqslant n$ f attains its maximum and minimum on $[x_{i-1}, x_i]$; thus for $1 \leqslant i \leqslant n$, there exist $M_i, m_i \in [x_{i-1}, x_i]$ such that

$$f(M_i) = \text{the maximum of } f \text{ on } [x_{i-1}, x_i]$$

and

$$f(m_i) = \text{the minimum of } f \text{ on } [x_{i-1}, x_i].$$

The sum $S(n) \equiv \sum\limits_{i=1}^{n} f(M_i)\Delta x$ is called the upper sum of f for the partition $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$, and $s(n) \equiv \sum\limits_{i=1}^{n} f(m_i)\Delta x$ is called the lower sum of f for the partition $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. By the definition of the upper sum and lower sum, we find that for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} f(m_i)\Delta x \leqslant \mathcal{A}(\mathrm{R}) \leqslant \sum_{i=1}^{n} f(M_i)\Delta x\,.$$

If the limits of the both sides exist and are identical as Δx approaches 0 (which is the same as n approaches infinity), by the Squeeze Theorem we can conclude that $\mathcal{A}(\mathrm{R})$ is the same as the limit.

If f is not continuous, then f might not attain its extrema on the interval $[x_{i-1}, x_i]$. In this case, it might be impossible to form the upper sum or the lower sum for a given partition. On the other hand, the left end-point rule $\sum\limits_{i=1}^{n} f(x_{i-1})\Delta x$ and the right end-point rule $\sum\limits_{i=1}^{n} f(x_i)\Delta x$ of approximating the area are always possible. We can even consider the "mid-point rule" approximation given by

$$\sum_{i=1}^{n} f\Big(\frac{x_{i-1} + x_i}{2}\Big)\Delta x$$

and consider the limit of the expression above as n approaches infinity.

In general, in order to find an approximation of $\mathcal{A}(\mathrm{R})$, the interval $[a, b]$ does not have to be divided into sub-intervals with equal length. Assume that $[a, b]$ are divided into n sub-intervals and the end-points of those sub-intervals are ordered as $a = x_0 < x_1 < x_2 <$

$\cdots < x_n = b$. The collection of end-points $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is called a ***partition*** of $[a, b]$. Then the "left end-point rule" approximation for the partition $\mathcal{P}$ is given by

$$\ell(\mathcal{P}) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

and the "right end-point rule" approximation for the partition $\mathcal{P}$ is given by

$$r(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})\,.$$

and the limit process as $n \to \infty$ in the discussion above is replaced by the limit process as the norm of partition $\mathcal{P}$, denoted by $\|\mathcal{P}\|$ and defined by $\|\mathcal{P}\| \equiv \max\big\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\big\}$, approaches 0. Before discussing what the limits above mean, let us look at the following examples.

Example 4.4. Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \leqslant x \leqslant 1$. Let $x_i = \dfrac{i^2}{n^2}$ and $\mathcal{P} = \{x_0 = 0 < x_1 < \cdots < x_n = 1\}$. We note that

$$\|\mathcal{P}\| = \max\Big\{\frac{i^2 - (i-1)^2}{n^2} \,\Big|\, 1 \leqslant i \leqslant n\Big\} = \max\Big\{\frac{2i-1}{n^2} \,\Big|\, 1 \leqslant i \leqslant n\Big\} = \frac{2n-1}{n^2}$$

thus $\|\mathcal{P}\| \to 0$ is equivalent to that $n \to \infty$.

Using the right end-point rule (which is the same as the upper sum),

$$\begin{aligned} S(\mathcal{P}) &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i}{n}\frac{2i-1}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - i) \\ &= \frac{1}{n^3}\Big[\frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2}\Big] \\ &= \frac{1}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big) - \frac{1}{2n}\Big(1+\frac{1}{n}\Big)\,; \end{aligned}$$

thus

$$\lim_{\|\mathcal{P}\|\to 0} S(\mathcal{P}) = \lim_{n\to\infty}\Big[\frac{1}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big) - \frac{1}{2n}\Big(1+\frac{1}{n}\Big)\Big] = \frac{2}{3}\,.$$

Using the left end-point rule (which is the same as the lower sum),

$$\begin{aligned} s(\mathcal{P}) &= \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i-1}{n}\frac{2i-1}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - 3i + 1) \\ &= \frac{1}{n^3}\Big[\frac{n(n+1)(2n+1)}{3} - \frac{3n(n+1)}{2} + n\Big] \\ &= \frac{1}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big) - \frac{3}{2n}\Big(1+\frac{1}{n}\Big) + \frac{1}{n^2}\,; \end{aligned}$$

thus

$$\lim_{\|\mathcal{P}\|\to 0} s(\mathcal{P}) = \lim_{n\to\infty} \Big[\frac{1}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big) - \frac{3}{2n}\Big(1+\frac{1}{n}\Big) + \frac{1}{n^2}\Big] = \frac{2}{3}\,.$$

Therefore, the area of the region of interest is $\dfrac{2}{3}$.

Example 4.5. In this example we use a different approach to compute $\mathcal{A}(\mathrm{R})$ in Example 4.2. Assume that $0 < a < b$. Let $r = \big(\dfrac{b}{a}\big)^{\frac{1}{n}}$, $x_i = ar^i$, and $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

Claim: If $c > 1$, then $c^{\frac{1}{n}} = 1$ as n approaches infinity.

Proof of the claim: If $c > 1$, then $c^{\frac{1}{n}} > 1$. Let $y_n = c^{\frac{1}{n}} - 1$. Then $c = (1+y_n)^n \geqslant 1 + ny_n$ which implies that $0 < y_n \leqslant \dfrac{c-1}{n}$ for all $n \in \mathbb{N}$. By the Squeeze Theorem, $c^{\frac{1}{n}} \to 1$ as $n \to \infty$. □

Note that the claim above implies that $r \to 1$ as $n \to \infty$. Moreover, $x_i - x_{i-1} = a(r^i - r^{i-1}) = ar^{i-1}(r-1)$; thus

$$0 < a(r-1) = x_1 - x_0 \leqslant \|\mathcal{P}\| = x_n - x_{n-1} = ar^{n-1}(r-1) < b(r-1)\,.$$

Therefore, $\|\mathcal{P}\| \to 0$ is equivalent to that $n \to \infty$.

Using the "left end-point rule" approximation of the area,

$$\begin{aligned}\mathcal{A}(\mathrm{R}) &= \lim_{n\to\infty}\sum_{i=1}^{n} x_{i-1}^2(x_i - x_{i-1}) = \lim_{n\to\infty}\sum_{i=1}^{n} a^2r^{2(i-1)}ar^{i-1}(r-1) = a^3\lim_{n\to\infty}(r-1)\sum_{i=1}^{n} r^{3(i-1)}\\ &= a^3\lim_{n\to\infty}(r-1)\frac{r^{3n}-1}{r^3-1} = a^3\lim_{n\to\infty}\frac{\frac{b^3}{a^3}-1}{r^2+r+1} = \frac{b^3-a^3}{3}\,.\end{aligned}$$

Similarly, when applying the "right end-point rule" approximation, we obtain that

$$\lim_{n\to\infty}\sum_{i=1}^{n} x_i^2(x_i - x_{i-1}) = a^3\lim_{n\to\infty}(r-1)\sum_{i=1}^{n} r^{3i} = a^3\lim_{n\to\infty}(r-1)\frac{r^{3n+3}-r^3}{r^3-1} = \frac{b^3-a^3}{3}\,.$$

This also gives the area of the region R.

To compute an approximated value of $\mathcal{A}(\mathrm{R})$, there is no reason for evaluating the function at the left end-points or the right end-points like what we have discussed above. For example, we can also consider the "mid-point rule"

$$m(\mathcal{P}) = \sum_{i=1}^{n} f\Big(\frac{x_i + x_{i-1}}{2}\Big)(x_i - x_{i-1})$$

to approximate the value of $\mathcal{A}(\mathrm{R})$, and compute the limit of the sum above as $\|\mathcal{P}\|$ approaches 0 in order to obtain $\mathcal{A}(\mathrm{R})$. In fact, we should be able to consider any point $c_i \in [x_{i-1}, x_i]$ and consider the limit of the sum

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

if the region R does have area.

Now let us forget about the concept of the area. For a general function $f : [a,b] \to \mathbb{R}$, we can also consider the limit above as $\|\mathcal{P}\|$ approaches 0, if the limit exists. The discussion above motivates the following definitions.

Definition 4.6: Partition of Intervals and Riemann Sums

A finite set $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is said to be a partition of the closed interval $[a,b]$ if $a = x_0 < x_1 < \cdots < x_n = b$. Such a partition $\mathcal{P}$ is usually denoted by $\{a = x_0 < x_1 < \cdots < x_n\}$. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number $\max\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\}$; that is,

$$\|\mathcal{P}\| \equiv \max\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\}.$$

A partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is called regular if $x_i - x_{i-1} = \|\mathcal{P}\|$ for all $1 \leqslant i \leqslant n$.

Let $f : [a,b] \to \mathbb{R}$ be a function. A Riemann sum of f for the the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ is a sum which takes the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}),$$

where the set $\Xi = \{c_0, c_1, \cdots, c_{n-1}\}$ satisfies that $x_{i-1} \leqslant c_i \leqslant x_i$ for each $1 \leqslant i \leqslant n$.

Definition 4.7: Riemann Integrals - 黎曼積分

Let $f : [a,b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a,b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A (is unique and) is called the Riemann integral of f on $[a,b]$ and is denoted by $\displaystyle\int_{[a,b]} f(x)\,dx$.

Remark 4.8. For conventional reason, the Riemann integral of f over the interval with left end-point a and right-end point b is written as $\displaystyle\int_a^b f(x)\,dx$, and is called the definite integral of f from a to b. The function f sometimes is called the integrand of the integral.

We also note that here in the representation of the integral, x is a dummy variable; that is, we can use any symbol to denote the independent variable; thus

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

and etc.