微積分 MA1001-A 上課筆記（精簡版）

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Definition 3.11

Let f be defined on an interval I.

1. f is said to be increasing on I if
$$f(x_1) \leqslant f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$
2. f is said to be decreasing on I if
$$f(x_1) \geqslant f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$
3. f is said to be strictly increasing on I if
$$f(x_1) < f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$
4. f is said to be strictly decreasing on I if
$$f(x_1) > f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

Theorem 3.15

Let $f : [a, b] \to \mathbb{R}$ be continuous and f is differentiable on (a, b).

1. If $f'(x) \geqslant 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
2. If $f'(x) \leqslant 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.
3. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.
4. If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.

Theorem 3.17: The First Derivative Test

Let f be a continuous function defined on an open interval I containing c. If f is differentiable on I, except possibly at c, then

1. If f' changes from negative to positive at c, then $f(c)$ is a local minimum of f.
2. If f' changes from positive to negative at c, then $f(c)$ is a local maximum of f.
3. If f' is sign definite on both sides of (and near) c, then $f(c)$ is neither a relative minimum or relative maximum of f.

Definition 3.19

Let f be differentiable on an open interval I. The graph of f is concave upward（凹向上）on I if f' is strictly increasing on the interval and concave downward（凹向下）on I if f' is strictly decreasing on the interval.

- Graphical interpretation of concavity: Let f be differentiable on an open interval I.

  1. If the graph of f is concave upward on I, then the graph of f lies above all of its tangent lines on I.

  2. If the graph of f is concave downward on I, then the graph of f lies below all of its tangent lines on I.

Theorem 3.21: Test for Concavity

Let f be a twice differentiable function on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 3.23: Point of inflection（反曲點）

Let f be a differentiable function on an open interval containing c. The point $\big(c, f(c)\big)$ is called a point of inflection (or simply an inflection point) of the graph of f if the concavity of f changes from upward to downward or downward to upward at this point.

Example 3.24. Recall Example 3.22 $\Big(f(x) = \dfrac{6}{x^2+3}$ with $f''(x) = \dfrac{36(x^2-1)}{(x^2+3)^3}\Big)$. Since f'' changes sign at $x = \pm 1$, $\Big(\pm 1, \dfrac{3}{2}\Big)$ are both points of inflection of the graph of f.

Theorem 3.25

Let f be a differentiable function on an open interval containing c. If $\big(c, f(c)\big)$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Remark 3.26. A point $\big(c, f(c)\big)$ may not be an inflection point of the graph of f even if $f''(c) = 0$. For example, the point $(0, 0)$ is not an inflection point of $f(x) = x^4$ since $f''(x) > 0$ for all $x \neq 0$ which implies that the concavity of f does not change at $c = 0$.

Example 3.27. Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$. Note that the zero of f'' is $x = 0$ or $x = 2$ (since $f''(x) = 12x^2 - 24x$). Since $f''(x) > 0$ if $x < 0$ or $x > 2$, and $f''(x) > 0$ if $0 < x < 2$, we find that $(0, 0)$ and $(2, -16)$ are points of inflection of the graph of f.

Theorem 3.28

Let f be a twice differentiable function on an open interval I containing c, and c is a critical point of f.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum of f on I.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum of f on I.

Remark 3.29. If $f''(c) = 0$ for some critical point c of f, then f may have a relative maximum, a relative minimum, or neither at c. In such cases, you should use the First Derivative Test.

Proof of Theorem 3.28. Since $f''(c) > 0$, there exist $\delta > 0$ such that

$$\left|\frac{f'(x) - f'(c)}{x - c} - f''(c)\right| < \frac{f''(c)}{2} \quad \text{if } 0 < |x - c| < \delta.$$

Since c is a critical point of f, $f'(c) = 0$; thus the inequality above implies that

$$\frac{f''(c)}{2} < \frac{f'(x)}{x - c} < \frac{3f''(c)}{2} \quad \text{if } 0 < |x - c| < \delta.$$

In particular,

$$\begin{aligned} 0 < \frac{f'(c)}{2}(x - c) < f'(x) \qquad & \text{if } 0 < x - c < \delta\,, \\ f'(x) < \frac{f'(c)}{2}(x - c) < 0 \qquad & \text{if } -\delta < x - c < 0\,. \end{aligned}$$

Therefore, f' changes from negative to positive at c; thus $f(c)$ is a relative minimum of f on I. □

Example 3.30. Recall Example 3.18 $\big(f(x) = \frac{1}{2}x - \sin x\big)$. We have established that $f\big(\frac{\pi}{3}\big)$ is a relative minimum of f on $(0, 2\pi)$ using the First Derivative Test. Note that $f''(x) = \sin x$; thus $f''\big(\frac{\pi}{3}\big) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} > 0$. Therefore, without using the First Derivative Test, we can still conclude that $f\big(\frac{\pi}{3}\big)$ is a relative minimum of f on $(0, 2\pi)$ by the second derivative test.

3.5 A Summary of Curve Sketching

When sketching the graph of functions, you need to have the following on the plot.

1. x-intercepts and y-intercepts;
2. asymptotes;
3. absolution extrema and relative extrema;
4. points of inflection.

Example 3.31. Sketch the graph of the function $f(x) = \dfrac{3x-2}{\sqrt{2x^2+1}}$.

First, we note that the x-intercepts and y-intercepts are $\left(\frac{3}{2}, 0\right)$ and $(0, f(0)) = (0, -2)$. To determine the asymptotes, since $\sqrt{2x^2+1}$ are never zero, there is no vertical asymptote. As for the horizontal and slant asymptotes, by the fact that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{x}} = \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}} = \lim_{y\to 0^+} \frac{3-2y}{\sqrt{2+y^2}} = \frac{3}{\sqrt{2}}$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x) = \lim_{x\to\infty} \frac{-3x-2}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{-3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}} = \lim_{y\to 0^+} \frac{3-2y}{-\sqrt{2+y^2}} = -\frac{3}{\sqrt{2}},$$

we find that there are two horizontal asymptotes $y = \pm\dfrac{3}{\sqrt{2}}$.

By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{3\sqrt{2x^2+1} - (3x-2)\frac{d}{dx}(2x^2+1)^{\frac{1}{2}}}{2x^2+1} = \frac{3\sqrt{2x^2+1} - (3x-2)\frac{1}{2}(2x^2+1)^{-\frac{1}{2}} \cdot (4x)}{2x^2+1} \\ &= \frac{3(2x^2+1) - 2x(3x-2)}{(2x^2+1)^{\frac{3}{2}}} = \frac{4x+3}{(2x^2+1)^{\frac{3}{2}}} \end{aligned}$$

and

$$\begin{aligned} f''(x) &= \frac{4(2x^2+1)^{\frac{3}{2}} - (4x+3)\frac{3}{2}(2x^2+1)^{\frac{1}{2}} \cdot (4x)}{(2x^2+1)^3} = \frac{4(2x^2+1) - 6x(4x+3)}{(2x^2+1)^{\frac{5}{2}}} \\ &= \frac{-16x^2-18x+4}{(2x^2+1)^{\frac{5}{2}}} = \frac{-2(8x^2+9x-2)}{(2x^2+1)^{\frac{5}{2}}}. \end{aligned}$$

Therefore, $x = -\frac{3}{4}$ is the only critical point and since f' changes from negative to positive at $-\frac{3}{4}$, $f\big(-\frac{3}{4}\big)$ is a relative minimum of f.

$f''(x) = 0$ occurs at $x_1 = \frac{-9-\sqrt{145}}{16}$ and $x_2 = \frac{-9+\sqrt{145}}{16}$. Since f'' changes sign at x_1 and x_2, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are inflection points of the graph of f.

3.6 Optimization Problems

Explanation of examples in Section 3.7 in the textbook:

1. 一製造商想設計一個底部為正方形、表面積 108 平方公分且上方有開口的箱子。要怎麼設計才能讓箱子容積最大？

2. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$? 拋物線 $y = 4 - x^2$ 上哪個點到 $(0, 2)$ 最近？

3. 試找出最小面積的方形頁面使之能上下留白三公分、左右留白兩公分且要包含 216 平方公分的長方形區域可用於印刷。

4. 兩根分別為 12 公尺及 28 公尺高的桿子相距 30 公尺。找出地面上一點使之到兩桿之頂端之距離和最小。

5. Four meters of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area? 一總長 4 公尺的線要被分為兩段用來圍出一個正方形和一個圓形。要怎麼分段才能圍出最大的面積。