# 微積分 MA1001-A 上課筆記（精簡版）

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**Theorem 2.13: Product Rule**

Let $f, g : (a, b) \to \mathbb{R}$ be real-valued functions, and $c \in (a, b)$. If $f$ and $g$ are differentiable at $c$, then $fg$ is differentiable at $c$ and

$$\frac{d}{dx}\bigg|_{x=c}(fg)(x) = f'(c)g(c) + f(c)g'(c)\,.$$

**Theorem 2.15: Quotient Rule**

Let $f, g : (a, b) \to \mathbb{R}$ be real-valued functions, and $c \in (a, b)$. If $f$ and $g$ are differentiable at $c$ and $g(c) \neq 0$, then $\frac{f}{g}$ is differentiable at $c$ and

$$\frac{d}{dx}\bigg|_{x=c}\frac{f}{g}(x) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}\,.$$

**Theorem 2.18: Chain Rule - 連鎖律**

Let $I, J$ be open intervals, $f : J \to \mathbb{R}$, $g : I \to \mathbb{R}$ be real-valued functions, and the range of $g$ is contained in $J$. If $g$ is differentiable at $c \in I$ and $f$ is differentiable at $g(c)$, then $f \circ g$ is differentiable at $c$ and

$$\frac{d}{dx}\bigg|_{x=c}(f \circ g)(x) = f'(g(c))g'(c)\,.$$

**How to memorize the chain rule?** Let $y = g(x)$ and $u = f(y)$. Then the derivative $u = (f \circ g)(x)$ is $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$.

**Definition 2.23**

Let $f$ be a function defined on an open interval $I$. $f$ is said to be continuously differentiable on $I$ if $f$ is differentiable on $I$ and $f'$ is continuous on $I$.

## 2.4 Implicit Differentiation

An implicit function is a function that is defined implicitly by an equation that $x$ and $y$ satisfy, by associating one of the variables (the value $y$) with the others (the arguments $x$). For example, $x^2 + y^2 = 1$ and $x = \cos y$ are implicit functions. Sometimes we know how to express $y$ in terms of $x$ from the equation (such as the first case above $y = \sqrt{1 - x^2}$

or $y = -\sqrt{1-x^2}$), while in most cases there is no way to know what the function $y$ of $x$ exactly is.

Given an implicit function (without solving for $y$ in terms of $x$ from the equation), can we find the derivative of $y$? This is the main topic of this section. We first focus on implicit functions of the form $f(x) = g(y)$. If $f(a) = g(b)$, we are interested in how the set $\{(x,y) \,|\, f(x) = g(y)\}$ looks like "mathematically" near $(a,b)$.

**Theorem 2.24: Implicit Function Theorem - 隱函數定理簡單版**

Let $f, g$ be continuously differentiable functions defined on some open intervals, and $f(a) = g(b)$. If $g'(b) \neq 0$, then there exists a unique continuously differentiable function $y = h(x)$, defined in an open interval containing $a$, satisfying that $b = h(a)$ and $f(x) = g(h(x))$.

**Example 2.25.** Let us compute the derivative of $h(x) = x^r$, where $r = \dfrac{p}{q}$ for some $p, q \in \mathbb{N}$ and $(p,q) = 1$. Write $y = h(x)$. Then $y^q = x^p$. Since $\dfrac{d}{dy}y^q = qy^{q-1} \neq 0$ if $y \neq 0$, by the Implicit Function Theorem we find that $h$ is differentiable at every $x$ satisfying $x \neq 0$. Since $h(x)^q = x^p$, by the chain rule we find that

$$qh(x)^{q-1}h'(x) = px^{p-1} \qquad \forall\, x \neq 0\,;$$

thus

$$h'(x) = \frac{p}{q}h(x)^{1-q}x^{p-1} = \frac{p}{q}x^{\frac{p}{q}(1-q)+p-1} = rx^{r-1} \qquad \forall\, x \neq 0\,.$$

If $r$ is a negative rational number, we can apply the quotient and find that

$$\frac{d}{dx}x^r = \frac{d}{dx}\frac{1}{x^{-r}} = \frac{rx^{-r-1}}{x^{-2r}} = rx^{r-1} \qquad \forall\, x \neq 0\,.$$

Therefore, we conclude that

$$\frac{d}{dx}x^r = rx^{r-1} \qquad \forall\, x \neq 0\,. \tag{2.4.1}$$

**Remark 2.26.** The derivative of $x^r$ can also be computed by first finding the derivative of $x^{\frac{1}{p}}$ $\Big($that is, find the limit $\displaystyle\lim_{\Delta x \to 0} \frac{(x+\Delta x)^{\frac{1}{p}} - x^{\frac{1}{p}}}{\Delta x}\Big)$ and then apply the chain rule.

**Example 2.27.** Suppose that $y$ is an implicit function of $x$ given that $y^3 + y^2 - 5y - x^2 = -4$.

1. Find $\dfrac{dy}{dx}$.

2. Find the tangent line passing through the point $(3, -1)$.

Let $f(x) = x^2 - 4$ and $g(y) = y^3 + y^2 - 5y$. Then $g'(y) = 3y^2 + 2y - 5$; thus if $y \neq 1$ or $y \neq -\dfrac{5}{3}$ $\big($or equivalently, $x \neq \pm 1$ or $x \neq \pm\sqrt{\dfrac{283}{27}}\big)$,

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}\,.$$

Since $(1, -3)$ satisfies the relation $y^3 + y^2 - 5y - x^2 = -4$, the slope of the tangent line passing through $(3, -1)$ is $\dfrac{2 \cdot 3}{3(-1)^2 + 2(-1) - 5} = -\dfrac{3}{2}$; thus the desired tangent line is

$$y = -\frac{3}{2}(x - 3) - 1\,.$$

**Example 2.28.** Find $\dfrac{dy}{dx}$ implicitly for the equation $\sin y = x$.

Let $f(x) = x$ and $g(y) = \sin y$. Then $g'(y) = \cos y$; thus if $y \neq n\pi + \dfrac{\pi}{2}$ (or equivalently, $x \neq \pm 1$),

$$\frac{dy}{dx} = \frac{1}{\cos y}\,.$$

There are, unfortunately, many implicit functions that are not given by the equation of the form $f(x) = g(y)$. Nevertheless, there is a more powerful version of the Implicit Function Theorem that guarantees the continuous differentiability of the implicit functions defined through complicated relations between $x$ and $y$ (written in the form $f(x, y) = 0$). In the following, we always assume that the implicit function given by the equation that $x$ and $y$ satisfy is differentiable.

**Example 2.29.** Find the second derivative of the implicit function given by the equation $y = \cos(5x - 3y)$.

Differentiate in $x$ once, we find that $\dfrac{dy}{dx} = -\sin(5x - 3y) \cdot \big(5 - 3\dfrac{dy}{dx}\big)$; thus

$$\frac{dy}{dx} = \frac{-5\sin(5x - 3y)}{1 - 3\sin(5x - 3y)} = \frac{5}{3}\Big[1 - \frac{1}{1 - 3\sin(5x - 3y)}\Big]\,. \tag{2.4.2}$$

Differentiate the equation above in $x$, we obtain that

$$\frac{d^2y}{dx^2} = -\frac{5}{3} \cdot \frac{3\cos(5x - 3y)(5 - 3y')}{\big[1 - 3\sin(5x - 3y)\big]^2} = -\frac{5\cos(5x - 3y)(5 - 3y')}{\big[1 - 3\sin(5x - 3y)\big]^2}$$

and (2.4.2) further implies that $\dfrac{d^2y}{dx^2} = -\dfrac{25\cos(5x - 3y)}{\big[1 - 3\sin(5x - 3y)\big]^3}$.

# Chapter 3. Applications of Differentiation

## 3.1 Extrema on an Interval

**Definition 3.1**

Let $f$ be defined on an interval $I$ containing $c$.

1. $f(c)$ is the minimum of $f$ on $I$ when $f(c) \leqslant f(x)$ for all $x$ in $I$.
2. $f(c)$ is the maximum of $f$ on $I$ when $f(c) \geqslant f(x)$ for all $x$ in $I$.

The minimum and maximum of a function on an interval are the extreme values, or extrema (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum, or the global minimum and global maximum, on the interval. Extrema can occur at interior points or end-points of an interval. Extrema that occur at the end-points are called end-point extrema.

**Theorem 3.2: Extreme Value Theorem - 極值定理**

If $f$ is continuous on a closed interval $[a, b]$, then $f$ has both a minimum and a maximum on the interval.（連續函數在閉區間上必有最大最小值）

**Definition 3.3**

Let $f$ be defined on an interval $I$ containing $c$.

1. If there is an open interval containing $c$ on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of $f$, or you can say that $f$ has a relative maximum at $\big(c, f(c)\big)$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a relative minimum of $f$, or you can say that $f$ has a relative minimum at $\big(c, f(c)\big)$.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called local maximum and local minimum, respectively.

**Definition 3.4**

Let $f$ be defined on an open interval containing $c$. The number/point $c$ is called a critical number or critical point of $f$ if $f'(c) = 0$ or if $f$ is not differentiable at $c$.