微積分 MA1001-A 上課筆記（精簡版）

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Chapter 2. Differentiation

2.1 The Derivatives of Functions

Definition 2.1

Let f be a function defined on an open interval containing c. If the limit $\lim\limits_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} = m$ exists, then the line passing through $\big(c, f(c)\big)$ with slope m is the tangent line to the graph of f at point $\big((c, f(c)\big)$.

Definition 2.2

Let f be a function defined on an open interval I containing c. f is said to be differentiable at c if the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}$$

exists. If the limit above exists, the limit is denoted by $f'(c)$ and called the derivative of f at c. When the derivative of f at each point of I exists, f is said to be differentiable on I and the derivative of f is a function denoted by f'.

Notation: If f is differentiable on an open interval I and $c \in I$, then we use the following notation:

$$f'(x) = \frac{d}{dx} f(x) = \frac{df(x)}{dx}, \qquad f'(c) = \frac{d}{dx}\Big|_{x=c} f(x).$$

Remark 2.3. Letting $x = c + \Delta x$ in the definition of the derivatives, then

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

if the limit exists.

Combining Example 2.4-2.6, we conclude that

$$\frac{d}{dx} x^n = \begin{cases} nx^{n-1} & \forall\, x \in \mathbb{R} \quad \text{if } n \in \mathbb{N} \cup \{0\}, \\ nx^{n-1} & \forall\, x \neq 0 \quad \text{if } n \in \mathbb{Z} \text{ and } n < 0. \end{cases} \tag{2.1.1}$$

我們注意到當 n 是負整數時，在計算 $\frac{d}{dx}\Big|_{x=c} x^n$ 時，已經必須先假設 $c \neq 0$ 才能計算導數，並非最後算出來 $\frac{d}{dx}\Big|_{x=c} x^n = nc^{n-1}$ 時發現 c 不可為零所以不能代入。這是一個非常重要的觀念！不能搞錯順序！

Example 2.7. Let $f(x) = \sin x$. By the sum and difference formula,

$$\begin{aligned} f(x+\Delta x) - f(x) &= \sin(x+\Delta x) - \sin x = \sin x \cos\Delta x + \sin\Delta x \cos x - \sin x \\ &= \sin x(\cos\Delta x - 1) + \sin\Delta x \cos x\,; \end{aligned}$$

thus by the fact that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$, we find that

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \Big[\sin x \frac{\cos\Delta x - 1}{\Delta x} + \frac{\sin\Delta x}{\Delta x}\cos x\Big] = \cos x\,. \tag{2.1.2}$$

In other words, the derivative of the sine function is cosine.

On the other hand, let $g(x) = \cos x$. Then $g(x) = -f\big(x - \frac{\pi}{2}\big)$. Then if $\Delta x \neq 0$,

$$\frac{g(x+\Delta x) - g(x)}{\Delta x} = -\frac{f\big(x - \frac{\pi}{2} + \Delta x\big) - f\big(x - \frac{\pi}{2}\big)}{\Delta x}\,;$$

thus

$$\lim_{\Delta x\to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = -\cos\big(x - \frac{\pi}{2}\big) = -\sin x\,.$$

In other words, the derivative of the cosine function is minus sine. To summarize,

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx}\cos x = -\sin x\,. \tag{2.1.3}$$

Remark 2.8. If f is a function defined on a interval I, and c is one of the end-point. Then it is possible to define the one-sided derivative. For example, if c is the left end-point of I, then we can consider the limit

$$\lim_{\Delta x\to 0^+} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c}$$

if it exists. The limit above, if exists, is called the derivatives of f at c from the right.

Theorem 2.9: 可微必連續

Let f be a function defined on an open interval I, and $c \in I$. If f is differentiable at c, then f is continuous at c.

Proof. If $x \neq c$, $f(x) - f(c) = \dfrac{f(x) - f(c)}{x - c}(x - c)$. Since the limit $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ exists and $\lim\limits_{x\to c}(x - c) = 0$, by Theorem 1.12 we conclude that

$$\lim_{x\to c} \big[f(x) - f(c)\big] = \Big(\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\Big)\Big(\lim_{x\to c}(x - c)\Big) = 0\,.$$

Therefore, $\lim\limits_{x\to c} f(x) = f(c)$ which shows that f is continuous at c. □

Remark 2.10. When f is continuous on an open interval I, f is **not** necessary differentiable on I. For example, consider $f(x) = |x|$. Then Theorem 1.12 implies that f is continuous on I, but $\lim\limits_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$ D.N.E.

2.2 Rules of Differentiation

Theorem 2.11

We have the following differentiation rules:

1. If k is a constant, then $\frac{d}{dx}k = 0$.
2. If n is a non-zero integer, then $\frac{d}{dx}x^n = nx^{n-1}$ (whenever x^{n-1} makes sense).
3. $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$.
4. If k is a constant and $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then kf is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[kf(x)\big] = kf'(c)\,.$$
5. If $f, g : (a, b) \to \mathbb{R}$ are differentiable at $c \in (a, b)$, then $f \pm g$ is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[f(x) \pm g(x)\big] = f'(c) \pm g'(c)\,.$$

Proof of 5. Let $h(x) = f(x) + g(x)$. Then if $\Delta x \neq 0$,

$$\frac{h(c + \Delta x) - h(c)}{\Delta x} = \frac{f(c + \Delta x) - f(c)}{\Delta x} + \frac{g(c + \Delta x) - g(c)}{\Delta x}\,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = f'(c) \qquad \text{and} \qquad \lim_{\Delta x \to 0} \frac{g(c + \Delta x) - g(c)}{\Delta x}$$

exist. Therefore, by Theorem 1.12,

$$h'(c) = f'(c) + g'(c)\,.$$

The conclusion for the difference can be proved in the same way. □

Example 2.12. Let $f(x) = 3x^2 - 5x + 7$. Then

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}(3x^2 - 5x) + \frac{d}{dx}7 = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x)\\ &= 3\frac{d}{dx}x^2 - 5\frac{d}{dx}x = 3\cdot(2x) - 5 = 6x - 5\,.\end{aligned}$$

In general, for a polynomial function

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \equiv \sum_{k=0}^{n} a_k x^k\,,$$

where $a_0, a_1, \cdots, a_n \in \mathbb{R}$, by induction we can show that

$$\frac{d}{dx}p(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1 = \sum_{k=1}^{n} ka_k x^{k-1}\,.$$

Theorem 2.13: Product Rule

Let $f, g : (a,b) \to \mathbb{R}$ be real-valued functions, and $c \in (a,b)$. If f and g are differentiable at c, then fg is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c}(fg)(x) = f'(c)g(c) + f(c)g'(c)\,.$$

Proof. Let $h(x) = f(x)g(x)$. Then

$$\begin{aligned}h(c+\Delta x) - h(c) &= f(c+\Delta x)g(c+\Delta x) - f(c)g(c)\\ &= f(c+\Delta x)g(c+\Delta x) - f(c)g(c+\Delta x) + f(c)g(c+\Delta x) - f(c)g(c)\\ &= \big[f(c+\Delta x) - f(c)\big]g(c+\Delta x) + f(c)\big[g(c+\Delta x) - g(c)\big]\,.\end{aligned}$$

Therefore, if $\Delta x \neq 0$,

$$\frac{h(c+\Delta x) - h(c)}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x}g(c+\Delta x) + f(c)\frac{g(c+\Delta x) - g(c)}{\Delta x}\,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x\to 0}\frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c)\,,\ \lim_{\Delta x\to 0}\frac{g(c+\Delta x) - g(c)}{\Delta x}\,,\ \text{and}\ \lim_{\Delta x\to 0}g(c+\Delta x) = g(c)$$

exist. By Theorem 1.12,

$$h'(c) = f'(c)g(c) + f(c)g'(c)$$

which concludes the product rule. □

Example 2.14. Let $f(x) = x^3 \sin x$. Then the product rule implies that

$$f'(x) = 3x^2 \sin x + x^3 \cos x\,.$$

Theorem 2.15: Quotient Rule

Let $f, g : (a, b) \to \mathbb{R}$ be real-valued functions, and $c \in (a, b)$. If f and g are differentiable at c and $g(c) \neq 0$, then $\dfrac{f}{g}$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c} \frac{f}{g}(x) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}\,.$$

Proof. Let $h(x) = \dfrac{f(x)}{g(x)}$. Then

$$\begin{aligned} h(c+\Delta x) - h(c) &= \frac{f(c+\Delta x)}{g(c+\Delta x)} - \frac{f(c)}{g(c)} = \frac{f(c+\Delta x)g(c) - f(c)g(c+\Delta x)}{g(c)g(c+\Delta x)} \\ &= \frac{f(c+\Delta x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(c+\Delta x)}{g(c)g(c+\Delta x)} \\ &= \frac{\big[f(c+\Delta x) - f(c)\big]g(c) - f(c)\big[g(c+\Delta x) - g(c)\big]}{g(c)g(c+\Delta x)}\,. \end{aligned}$$

Therefore, if $\Delta x \neq 0$,

$$\frac{h(c+\Delta x) - h(c)}{\Delta x} = \frac{1}{g(c)g(c+\Delta x)}\Big[\frac{f(c+\Delta x) - f(c)}{\Delta x}g(c) - f(c)\frac{g(c+\Delta x) - g(c)}{\Delta x}\Big]\,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c)\,,\ \lim_{\Delta x \to 0} \frac{g(c+\Delta x) - g(c)}{\Delta x}\,,\ \text{ and } \lim_{\Delta x \to 0} g(c+\Delta x) = g(c)$$

exist. By Theorem 1.12,

$$h'(c) = \frac{1}{g(c)^2}\Big[f'(c)g(c) - f(c)g'(c)\Big]$$

which concludes the quotient rule. □

Example 2.16. Let n be a positive integer and $f(x) = x^{-n}$. We have shown by definition that $f'(x) = -nx^{-n-1}$ if $x \neq 0$. Now we use Theorem 2.15 to compute the derivative of f: if $x \neq 0$,

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = -\frac{\frac{d}{dx}x^n}{x^{2n}} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}\,.$$

Example 2.17. Since $\tan x = \dfrac{\sin x}{\cos x}$, by Theorem 2.15 we have

$$\frac{d}{dx}\tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\,.$$

Similarly, we also have

$$\begin{aligned}
\frac{d}{dx}\cot x &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x\,,\\
\frac{d}{dx}\sec x &= -\frac{-\sin x}{\cos^2 x} = \sec x \tan x\,,\\
\frac{d}{dx}\csc x &= -\frac{\cos x}{\sin^2 x} = -\cot x \csc x\,.
\end{aligned}$$

• **Higher-order derivatives**:

Let f be defined on an open interval $I = (a, b)$. If f' exists on I and possesses derivatives at every point in I, by definition we use f'' to denote the derivative of f'.In other words,

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\frac{d}{dx}f(x) \equiv \frac{d^2}{dx^2}f(x) = \frac{d^2 f(x)}{dx^2}\Big(= \frac{d^2 y}{dx^2} \text{ if } y = f(x)\Big)\,.$$

The function f'' is called the second derivative of f. Similar as the "first" derivative case, $f''(c) = \dfrac{d^2}{dx^2}\Big|_{x=c} f(x)$.

The third derivatives and even higher-order derivatives are denoted by the following: if $y = f(x)$,

$$\begin{array}{rcccc}
\text{Third derivative: } & y''' & f'''(x) & \dfrac{d^3}{dx^3}f(x) & \dfrac{d^3 f(x)}{dx^3}\\
\text{Fourth derivative: } & y^{(4)} & f^{(4)}(x) & \dfrac{d^4}{dx^4}f(x) & \dfrac{d^4 f(x)}{dx^4}\\
 & & \vdots & & \\
\text{n-th derivative: } & y^{(n)} & f^{(n)}(x) & \dfrac{d^n}{dx^n}f(x) & \dfrac{d^n f(x)}{dx^n}\,.
\end{array}$$