微積分 MA1001-A 上課筆記（精簡版）

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1.3 Infinite Limits and Asymptotes

Definition 1.48

Let f be defined on an open interval containing c (except possible at c). The statement

$$\lim_{x\to c} f(x) = \infty \,,$$

read "$f(x)$ approaches infinity as x approaches c", means that for every $N > 0$ there exists $\delta > 0$ such that

$$f(x) > N \quad \text{if} \quad 0 < |x - c| < \delta.$$

The statement

$$\lim_{x\to c} f(x) = \infty \,,$$

read "$f(x)$ approaches minus infinity as x approaches c", means that for every $N > 0$ there exists $\delta > 0$ such that

$$f(x) < -N \quad \text{if} \quad 0 < |x - c| < \delta.$$

To define the infinite limit from the left/right, replace $0 < |x - c| < \delta$ by $c < x < c + \delta / c - \delta < x < c$. To define the infinite limit as $x \to \infty / x \to -\infty$, replace $0 < |x - c| < \delta$ by $x > \delta / x < -\delta$.

Note that the statement $\lim_{x\to c} f(x) = \infty$ does **not** mean that the limit exists. It is a simple notation for saying that the value of f becomes unbounded as x approaches c and the limit fail to exist.

Definition 1.51: Vertical Asymptotes - 垂直漸近線

If f approaches infinity (or minus infinity) as x approaches c from the left or from the right, then the line $x = c$ is called a vertical asymptote of the graph of f.

Definition 1.52: Horizontal and Slant (Oblique) Asymptotes - 水平與斜漸近線

The straight line $y = mx + k$ is an asymptote of the graph of the function $y = f(x)$ if

$$\lim_{x\to\infty} \big[f(x) - mx - k\big] = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} \big[f(x) - mx - k\big] = 0 \,.$$

The straight line $y = mx + k$ is called a horizontal asymptote of the graph of f if $m = 0$, and is called a slant (oblique) asymptote of the graph of f if $m \neq 0$.

Theorem 1.55

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function $h(x) = \dfrac{f(x)}{g(x)}$ has a vertical asymptote at $x = c$.

Example 1.56. Let $f(x) = \tan x$. Note that $\tan x = \dfrac{\sin x}{\cos x}$. For $n \in \mathbb{Z}$, $\sin\left(n\pi + \dfrac{\pi}{2}\right) \neq 0$ and $\cos\left(n\pi + \dfrac{\pi}{2}\right) = 0$. Moreover, $\cos x \neq 0$ for every x in the open interval $\left(n\pi + \dfrac{\pi}{4}, n\pi + \dfrac{3\pi}{4}\right)$ except $n\pi + \dfrac{\pi}{2}$. Therefore, by the theorem above we find that $x = n\pi + \dfrac{\pi}{2}$ is a vertical asymptote of the graph of the tangent function for all $n \in \mathbb{Z}$.

Theorem 1.57

If $y = mx + k$ is a slant asymptote of the graph of the function $y = f(x)$, then

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad \text{or} \qquad m = \lim_{x\to-\infty} \frac{f(x)}{x}$$

and

$$k = \lim_{x\to\infty} \big[f(x) - mx\big] \qquad \text{of} \qquad k = \lim_{x\to-\infty} \big[f(x) - mx\big] \,.$$

Proof. It suffices to shows that $m = \displaystyle\lim_{x\to\infty} \frac{f(x)}{x}$ or $m = \displaystyle\lim_{x\to-\infty} \frac{f(x)}{x}$. W.L.O.G., we assume that $\displaystyle\lim_{x\to\infty} \big[f(x) - mx - k\big] = 0$. Then

$$\lim_{x\to\infty} \frac{f(x) - mx - k}{x} = 0\,.$$

On the other hand, $\displaystyle\lim_{x\to\infty} \frac{mx+k}{x} = m$. By the fact that $\dfrac{f(x)}{x} = \dfrac{f(x) - mx - k}{x} + \dfrac{mx+k}{x}$, we find that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x}$ exists and

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \left[\frac{f(x) - mx - k}{x}\right] + \lim_{x\to\infty} \frac{mx+k}{x} = m\,. \qquad \square$$

Chapter 2. Differentiation

2.1 The Derivatives of Functions

Definition 2.1

Let f be a function defined on an open interval containing c. If the limit $\displaystyle\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$ exists, then the line passing through $\big(c, f(c)\big)$ with slope m is the tangent line to the graph of f at point $\big((c, f(c)\big)$.

Definition 2.2

Let f be a function defined on an open interval I containing c. f is said to be differentiable at c if the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

exists. If the limit above exists, the limit is denoted by $f'(c)$ and called the derivative of f at c. When the derivative of f at each point of I exists, f is said to be differentiable on I and the derivative of f is a function denoted by f'.

Notation: The prime notation $'$ is associated with a function (of one variable) and is used to denote the derivative of that function. For a given function f defined on an open interval I and x being the name of the variable, the limit operation

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is denoted by $\dfrac{d}{dx} f(x)$ $\Big($or $\dfrac{df(x)}{dx}$ or even $\dfrac{dy}{dx}$ if $y = f(x)\Big)$, and the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta) - f(c)}{\Delta x}$$

is denoted by $\dfrac{d}{dx}\Big|_{x=c} f(x)$ but not $\dfrac{d}{dx} f(c)$ $\big(\dfrac{d}{dx} f(c)$ is in fact $0\big)$. The operator $\dfrac{d}{dx}$ is a differential operator called the differentiation and is applied to functions of variable x. However, for historical (and convenient) reason, $\dfrac{d}{dx} f(x)$ is sometimes denoted by $(f(x))'$ (so that $'$ is treated as the differential operator $\dfrac{d}{dx}$) and f' is sometimes denoted by $\dfrac{df}{dx}$ (so that f is always treated as a function of variable x).

Remark 2.3. Letting $x = c + \Delta x$ in the definition of the derivatives, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

if the limit exists.

Example 2.4. Let f be a constant function. Then f' is the zero function.

Example 2.5. Let $f(x) = x^n$, where n is a positive integer. Then

$$f(x + \Delta x) = x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n\,;$$

thus if $\Delta x \neq 0$,

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = nx^{n-1} + C^n_2 x^{n-2}\Delta x + \cdots + C^n_{n-1} x(\Delta x)^{n-2} + (\Delta x)^{n-1}\,.$$

The limit on the right-hand side is clearly nx^{n-1}, so we establish that

$$\frac{d}{dx} x^n = nx^{n-1}\,.$$

Example 2.6. Now suppose that $f(x) = x^{-n}$, where n is a positive integer. Then if $x + \Delta x \neq 0$,

$$f(x + \Delta x) = \frac{1}{x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n}\,;$$

thus if $x \neq 0$, $\Delta x \neq 0$, and $x + \Delta x \neq 0$ (which can be achieved if $|\Delta x| \ll 1$),

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-\big[C^n_1 x^{n-1} + C^n_2 x^{n-2}\Delta x + \cdots + C^n_{n-1} x(\Delta x)^{n-2} + (\Delta x)^{n-1}\big]}{x^n\big[x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n\big]}\,.$$

Therefore, if $x \neq 0$,

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

which shows $\frac{d}{dx} x^{-n} = -nx^{-n-1}$.

Combining the previous three examples, we conclude that

$$\frac{d}{dx} x^n = \begin{cases} nx^{n-1} & \forall\, x \in \mathbb{R} \quad \text{if } n \in \mathbb{N} \cup \{0\}\,, \\ nx^{n-1} & \forall\, x \neq 0 \quad \text{if } n \in \mathbb{Z} \text{ and } n < 0\,. \end{cases} \tag{2.1.1}$$