微積分 MA1001-A 上課筆記（精簡版）

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1.3 Continuity of Functions

Definition 1.34

Let f be a function defined on an interval I, and $c \in I$.

1. f is said to be right-continuous at c (or continuous from the right at c) if
$$\lim_{x \to c^+} f(x) = f(c) \, .$$
2. f is said to be left-continuous at c (or continuous from the left at c) if
$$\lim_{x \to c^-} f(x) = f(c) \, .$$
3. If c is the left end-point of I, f is said to be continuous at c if f is right-continuous at c.
4. If c is the right end-point of I, f is said to be continuous at c if f is left-continuous at c.
5. If c is an interior point of I; that is, c is neither the left end-point nor the right end-point of I, then f is said to be continuous at c if $\lim\limits_{x \to c} f(x) = f(c)$.

f is said to be discontinuous at c if f is not continuous at c, and in this case c is called a point of discontinuity (or simply a discontinuity) of f. f is said to be continuous (or a continuous function) on I if f is continuous at each point of I.

Remark 1.40. Let I be an interval, $c \in I$, and $f : I \to \mathbb{R}$ be a function. The continuity of f at c is equivalent to that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{if} \quad |x - c| < \delta \text{ and } x \in I \, .$$

Proposition 1.42

Let f, g be defined on an interval I, $c \in I$, and f, g be continuous at c. Then

1. $f \pm g$ is continuous at c.
2. fg is continuous at c.
3. $\dfrac{f}{g}$ is continuous at c if $g(c) \neq 0$.

Corollary 1.43

Let f, g be continuous functions on an interval I. Then

1. $f \pm g$ is continuous on I.
2. fg is continuous on I
3. $\dfrac{f}{g}$ is continuous (on its domain).

Theorem 1.44

Let I, J be open intervals,, $g : I \to \mathbb{R}$, $f : J \to \mathbb{R}$ be functions, and J contains the range of g. If g is continuous at c, then $f \circ g$ is continuous at c.

Proof. Let $\varepsilon > 0$ be given. Since f is continuous at $g(c)$, there exists $\delta_1 > 0$ such that

$$\big|f(x) - f\big(g(c)\big)\big| < \varepsilon \quad \text{if} \quad \big|x - g(c)\big| < \delta \text{ and } x \in J\,.$$

For such a δ_1, by the continuity of g at c there exists $\delta > 0$ such that

$$\big|g(x) - g(c)\big| < \delta_1 \quad \text{if} \quad |x - c| < \delta \text{ and } x \in I\,.$$

Therefore, if $|x - c| < \delta$ and $x \in I$, by the condition that J contains the range of g,

$$\big|g(x) - g(c)\big| < \delta_1 \text{ and } x \in J\,;$$

thus if $|x - c| < \delta$ and $x \in I$,

$$\big|f(g(x)) - f(g(c))\big| < \varepsilon$$

which shows the continuity of $f \circ g$ at c. □

Corollary 1.45

Let I, J be open intervals, and $g : I \to \mathbb{R}$, $f : J \to \mathbb{R}$ be continuous functions. If J contains the range of g, then $f \circ g$ is continuous on I.

Example 1.44. Let g be continuous on an interval I, and n be a positive integer. We show that g^n and $|g|^{\frac{1}{n}}$ are also continuous on I. Note that g^n is the function given by $g^n(x) = g(x)^n$ and $|g|^{\frac{1}{n}}$ is the function given by $|g|^{\frac{1}{n}} = |g(x)|^{\frac{1}{n}}$.

1. Let $f(x) = x^n$. Then Theorem 1.15 implies that f is continuous on $\mathbb{R}$. Since $\mathbb{R}$ contains the range of g, by the corollary above we find tat $f \circ g (\equiv g^n)$ is continuous on I.

2. Let $h(x) = |x|$. Then Theorem 1.12 implies that h is continuous on $\mathbb{R}$. Since $\mathbb{R}$ contains the range of g, by the corollary above we find that $h \circ g (\equiv |g|)$ is continuous on I.

   Let $f(x) = x^{\frac{1}{n}}$. Then Example 1.24 implies that f is continuous on the non-negative real axis $[0, \infty)$. Since $[0, \infty)$ contains the range of $|g|$, the corollary above shows that $f \circ |g| (\equiv |g|^{\frac{1}{n}})$ is continuous on I.

Theorem 1.45: Intermediate Value Theorem - 中間值定理

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Example 1.46 (Bisection method of finding zeros of continuous functions)**.** Let f be a function and $f(a)f(b) < 0$. Then the intermediate value theorem implies that there exists a zero c of f between a and b. How do we "find" (one of) this c? Consider the middle point $\dfrac{a+b}{2}$ of a and b. If $f\big(\dfrac{a+b}{2}\big) = 0$, then we find this zero, or otherwise we either have (1) $f(a)f\big(\dfrac{a+b}{2}\big) < 0$ or (2) $f(b)f\big(\dfrac{a+b}{2}\big) < 0$, and only one of them can happen. In either case we can consider the middle point of the two points at which the value of f have different sign. Continuing this process, we can locate one zero as accurate as possible.

Example 1.47. Let $f : [0, 1] \to [0, 1]$ be a continuous function. In the following we prove that there exists $c \in [0, 1]$ such that $f(c) = c$. To see this, W.L.O.G. we assume that $f(0) \neq 0$ and $f(1) \neq 1$ for otherwise we find c (which is 0 or 1) such that $f(c) = c$.

Define $g(x) = f(x) - x$. Then g is continuous (by Proposition 1.42). Since $f : [0, 1] \to [0, 1]$, $f(0) \neq 0$ and $f(1) \neq 1$, we must have $g(0) > 0$ and $g(1) < 0$. By the intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$, and this implies that there exists $c \in (0, 1)$ such that $f(c) = c$. So either (1) $f(0) = 0$, (2) $f(1) = 1$, or (3) there ia $c \in (0, 1)$ such that $f(c) = c$.

1.4 Infinite Limits and Asymptotes

Definition 1.48

Let f be defined on an open interval containing c (except possible at c). The statement

$$\lim_{x\to c} f(x) = \infty\,,$$

read "$f(x)$ approaches infinity as x approaches c", means that for every $N > 0$ there exists $\delta > 0$ such that

$$f(x) > N \quad \text{if} \quad 0 < |x - c| < \delta.$$

The statement

$$\lim_{x\to c} f(x) = \infty\,,$$

read "$f(x)$ approaches minus infinity as x approaches c", means that for every $N > 0$ there exists $\delta > 0$ such that

$$f(x) < -N \quad \text{if} \quad 0 < |x - c| < \delta.$$

To define the infinite limit from the left/right, replace $0 < |x - c| < \delta$ by $c < x < c + \delta / c - \delta < x < c$. To define the infinite limit as $x \to \infty / x \to -\infty$, replace $0 < |x - c| < \delta$ by $x > \delta / x < -\delta$.

Note that the statement $\lim_{x\to c} f(x) = \infty$ does **not** mean that the limit exists. It is a simple notation for saying that the value of f becomes unbounded as x approaches c and the limit fail to exist.

Example 1.49. $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$, $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$, and $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$.

Example 1.50. Later we will talk about the exponential function in detail. In the mean time, assume that you know the graph of $y = 2^x$. Then $\lim_{x\to\infty} 2^x = \infty$ and $\lim_{x\to-\infty} 2^x = 0$.

• **Asymptotes**（漸近線）: If the distance between the graph of a function and some fixed straight line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an asymptote of the graph.

Definition 1.51: Vertical Asymptotes - 垂直漸近線

If f approaches infinity (or minus infinity) as x approaches c from the left or from the right, then the line $x = c$ is called a vertical asymptote of the graph of f.

Definition 1.52: Horizontal and Slant (Oblique) Asymptotes - 水平與斜漸近線

The straight line $y = mx + k$ is an asymptote of the graph of the function $y = f(x)$ if

$$\lim_{x\to\infty} \big[f(x) - mx - k\big] = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} \big[f(x) - mx - k\big] = 0\,.$$

The straight line $y = mx + k$ is called a horizontal asymptote of the graph of f if $m = 0$, and is called a slant (oblique) asymptote of the graph of f if $m \neq 0$.

By the definition of horizontal asymptotes, it is clear that if $\lim\limits_{x\to\infty} f(x) = k$ or $\lim\limits_{x\to-\infty} f(x) = k$, then $y = k$ is a horizontal asymptote of the graph of f.

Example 1.53. Let $f(x) = \dfrac{x^2+3}{3x^2-4x+5}$. Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \dfrac{1}{3}$; thus $y = \dfrac{1}{3}$ is a horizontal asymptote of the graph of f.

Example 1.54. Let $f(x) = \dfrac{x^3+3}{3x^2-4x+5}$. Then $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$; thus the graph of f has no horizontal asymptote. However,

$$\lim_{x\to\infty} \Big[f(x) - \frac{x}{3}\Big] = \lim_{x\to\infty} \Big[\frac{3x^3+9}{3(3x^2-4x+5)} - \frac{x(3x^2-4x+5)}{3(3x^2-4x+5)}\Big] = \lim_{x\to\infty} \frac{4x^2-5x+9}{3(3x^2-4x+5)} = \frac{4}{9}\,;$$

thus $\lim\limits_{x\to\infty} \Big[f(x) - \dfrac{x}{3} - \dfrac{4}{9}\Big] = 0$. Therefore, $y = \dfrac{x}{3} + \dfrac{4}{9}$ is a slant asymptote of the graph of f.