微積分 MA1001-A 上課筆記（精簡版）

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Chapter 7

Applications of Integration

7.1 Area of a Region between Two Curves

The motivation of integration of functions is finding areas. Let us recall that if $f : [a, b] \to \mathbb{R}$ is non-negative, then the area A of the region bounded by the graph of f, the x-axis and vertical lines $x = a$ and $x = b$ is the integral of f on $[a, b]$ or in notation,

$$A = \int_a^b f(x)\, dx\,.$$

The idea above can be extended to the following statement: Let $f, g : [a, b] \to \mathbb{R}$ be continuous and $g(x) \leqslant f(x)$ for all $x \in [a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b \big[f(x) - g(x)\big]\, dx\,.$$

- **How about if the graphs of two continuous functions intersect?**

Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions but neither $g(x) \leqslant f(x)$ for all $x \in [a, b]$ nor $f(x) \leqslant g(x)$ for all $x \in [a, b]$. In other words, the graphs of f and g intersect (and transverse). In this case, the area of the region bounded by the graphs of f and g, as well as the vertical lines $x = a$ and $x = b$, is given by

$$A = \int_a^b \big|f(x) - g(x)\big|\, dx\,.$$

To find this integral, in general we need to find all the zeros of the function $h(x) = f(x)-g(x)$ and write the integral as sum of integrals on sub-intervals. To be more precise, suppose that the distinct zeros of h is given by $\{c_k\}_{k=1}^n$, where $a \leqslant c_1 < c_2 < \cdots < c_n \leqslant b$, then

$$\begin{aligned} A &= \int_a^b \big|f(x) - g(x)\big|\,dx \\ &= \int_a^{c_1} \big|f(x) - g(x)\big|\,dx + \sum_{k=1}^{n} \int_{c_{k-1}}^{c_k} \big|f(x) - g(x)\big|\,dx + \int_{c_n}^{b} \big|f(x) - g(x)\big|\,dx \\ &= \Big|\int_a^{c_1} \big[f(x) - g(x)\big]\,dx\Big| + \sum_{k=1}^{n} \Big|\int_{c_{k-1}}^{c_k} \big[f(x) - g(x)\big]\,dx\Big| + \Big|\int_{c_n}^{b} \big[f(x) - g(x)\big]\,dx\Big|. \end{aligned}$$

When f, g are continuous function on $\mathbb{R}$ and $h = f - g$ has finitely many distinct zeros $\{c_k\}_{k=1}^n$, we can also talk about the area of the (bounded) region bounded by the graph of f and g. This area is given by

$$A = \sum_{k=1}^{n} \Big|\int_{c_{k-1}}^{c_k} \big[f(x) - g(x)\big]\,dx\Big|$$

7.2 Volume: The Disk Method（圓盤法）

In the following two sections, the main focus is to develop ways of finding the volume of the so-called solids of revolution（旋轉體）, a solid formed by revolving a certain region about a line called the axis of revolution (and usually a line parallel to the x-axis or y-axis).

Example 7.1. The ball centered at the origin with radius r (usually denoted by $B(0,r)$ or $B_r(0)$), is a solid of revolution. It can be formed by revolving the region

$$\mathrm{R} = \Big\{(x,y)\,\Big|\,0 \leqslant y \leqslant \sqrt{r^2 - x^2}\Big\}$$

about the x-axis.

Example 7.2. A solid torus can be formed by revolving a disk

$$\mathrm{D} = \Big\{(x,y)\,\Big|\,(x-a)^2 + y^2 = r^2\Big\} \qquad \text{(where } 0 < a < r\text{)}$$

about the y-axis.

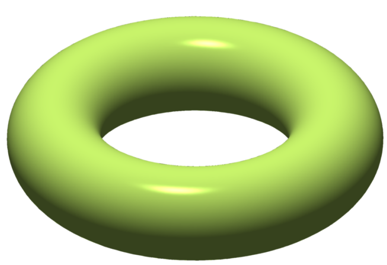


Figure 7.1: A solid torus

Consider the volume of a solid D formed by revolving a region R about the line $y = y_0$, where the region R is given by

$$\mathrm{R} = \left\{(x, y) \,\middle|\, x \in [a, b]\,, y_0 \leqslant y \leqslant f(x)\right\}$$

for some continuous function $f : [a, b] \to \mathbb{R}$ with $\min\limits_{x \in [a,b]} f(x) \geqslant y_0$. Note that the function $y = \pi\big[f(x) - c\big]^2$ is also continuous on $[a, b]$ thus integrable on $[a, b]$.

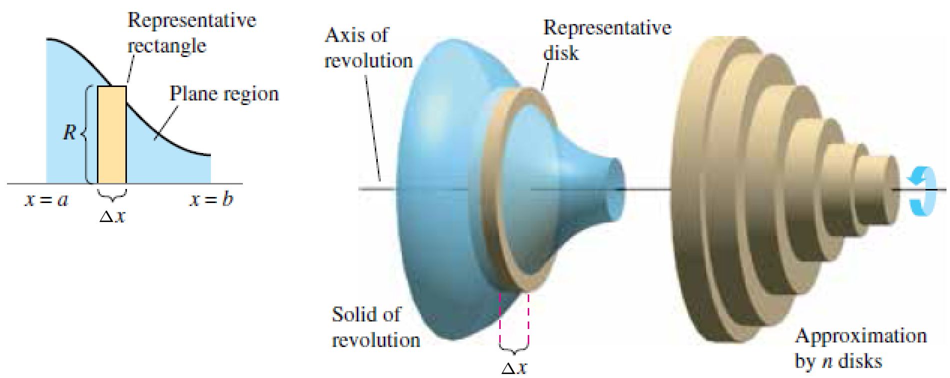


Figure 7.2: Disk method

Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$, and $\Delta x_i = x_i - x_{i-1}$. Then the volume of D is approximated by

$$\sum_{i=1}^{n} \pi\big[f(\xi_i) - y_0\big]^2 \Delta x_i\,,$$

where $\xi_i \in [x_{i-1}, x_i]$ for each $1 \leqslant i \leqslant n$. Note that the sum above is a Riemann sum of the function $y = \pi\big[f(x) - y_0\big]^2$ for partition $\mathcal{P}$.

When $\|\mathcal{P}\|$ approaches 0, we expect that the sum above approaches the volume of D. Since f is continuous on $[a, b]$, the function $y = \pi\big[f(x) - y_0\big]^2$ is Riemann integrable on $[a, b]$; thus for any given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|P\| < \delta$, then any Riemman sum of the function $y = \pi\big[f(x) - y_0\big]^2$ for $\mathcal{P}$ lies in the interval

$$\Big(\int_a^b \pi\big[f(x) - y_0\big]^2\,dx - \varepsilon,\ \int_a^b \pi\big[f(x) - y_0\big]^2\,dx + \varepsilon\Big)\,.$$

In particular, if $\max\left\{x_i - x_{i-1} \,\middle|\, 1 \leqslant i \leqslant n\right\} < \delta$,

$$\Big|\sum_{i=1}^{n} \pi\big[f(\xi_i) - y_0\big]^2 \Delta x_i - \int_a^b \pi\big[f(x) - y_0\big]^2\,dx\Big| < \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the volume of D can be computed by

$$\pi \int_a^b \big[f(x) - y_0\big]^2\, dx\,.$$

Example 7.3. The volume of the ball $B(0, r)$ is given by

$$\pi \int_{-r}^{r} \big(\sqrt{r^2 - x^2}\big)^2\, dx = \pi \int_{-r}^{r} (r^2 - x^2)\, dx = \pi \Big[r^2 x - \frac{1}{3}x^3\Big]\Big|_{x=-r}^{x=r} = \frac{4}{3}\pi r^3\,.$$

Example 7.4. The volume of the solid formed by revolving the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$ is given by

$$\begin{aligned}\pi \int_{-1}^{1} \big[(2 - x^2) - 1\big]^2\, dx &= \pi \int_{-1}^{1} (1 - x^2)^2\, dx = \pi \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx \\ &= \pi \Big[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\Big]\Big|_{x=-1}^{x=1} = \frac{16\pi}{15}\,.\end{aligned}$$

Similarly, if D is a solid formed by revolving a region R about the line $x = x_0$, where R is given by

$$\mathrm{R} = \big\{(x, y)\,\big|\, y \in [c, d]\,, x_0 \leqslant x \leqslant g(y)\big\}$$

for some continuous function $g : [c, d] \to \mathbb{R}$ with $\min\limits_{y\in[c,d]} g(y) \geqslant x_0$, then the volume of D is

$$\pi \int_c^d \big[g(y) - x_0\big]^2\, dy\,.$$

A solid of revolution may be formed by revolving a region away from the axis of revolution. In this case, the solid will have holes and the volume of

Suppose that the region R is given by

$$\mathrm{R} = \big\{(x, y)\,\big|\, a \leqslant x \leqslant b\,, y_0 \leqslant g(x) \leqslant y \leqslant f(x)\big\}\,,$$

where $f, g : [a, b] \to \mathbb{R}$ are continuous functions with $\max\limits_{x\in[a,b]} g(x) \leqslant \min\limits_{x\in[a,b]} f(x)$. Let R_1 and R_2 be given by

$$\mathrm{R}_1 = \big\{(x, y)\,\big|\, a \leqslant x \leqslant b\,, y_0 \leqslant y \leqslant f(x)\big\} \quad \text{and} \quad \mathrm{R}_2 = \big\{(x, y)\,\big|\, a \leqslant x \leqslant b\,, y_0 \leqslant y \leqslant g(x)\big\}\,.$$

Then the volume of the solid formed by revolving R about the line $y = y_0$ is the volume of the solid formed by revolving R_1 about the line $y = y_0$ minus the volume of the solid formed by revolving R_2 about the line $y = y_0$ and is given by

$$\pi \int_a^b \big[(f(x) - y_0)^2 - (g(x) - y_0)^2\big]\, dx\,.$$

Similarly, if R is given by

$$\mathrm{R} = \left\{(x,y)\,\middle|\, c \leqslant y \leqslant d\,, x_0 \leqslant g(y) \leqslant x \leqslant f(y)\right\},$$

where $f,g : [c,d] \to \mathbb{R}$ are continuous functions with $\max\limits_{y\in[c,d]} g(y) \leqslant \min\limits_{y\in[c,d]} f(y)$.. Then the volume of the solid formed by revolving R about the line $x = x_0$ is given by

$$\pi \int_c^d \left[(f(y)-x_0)^2 - (g(y)-x_0)^2\right] dy\,.$$

Example 7.5. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.

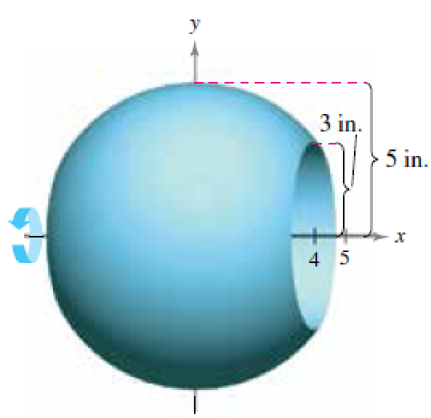
The points of intersection of the graphs of the two functions are $x = 0$ and $x = 1$, and $0 \leqslant x^2 \leqslant x$ on $[0,1]$. Therefore, the volume of the solid described above is given by

$$\pi \int_0^1 \left[\sqrt{x}^2 - (x^2)^2\right] dx = \pi \int_0^1 \left(x - x^4\right) dx = \pi\left(\frac{1}{2}x^2 - \frac{1}{5}x^4\right)\Big|_{x=0}^{x=1} = \frac{3\pi}{10}\,.$$

Example 7.6. The volume of the solid torus given in Example 7.2 is given by

$$\pi \int_{-r}^{r} \left[(a+\sqrt{r^2-y^2}-0)^2 - (a-\sqrt{r^2-y^2}-0)^2\right] dy$$
$$= 4a\pi \int_{-r}^{r} \sqrt{r^2-y^2}\, dy = 4a\pi \cdot \frac{\pi r^2}{2} = 2\pi^2 a r^2\,.$$

Example 7.7. Find the volume of the solid formed by a ball with 5 inch radius having a cylindrical hole as shown in the following figure.



Solid of revolution

The volume of the solid described above is given by

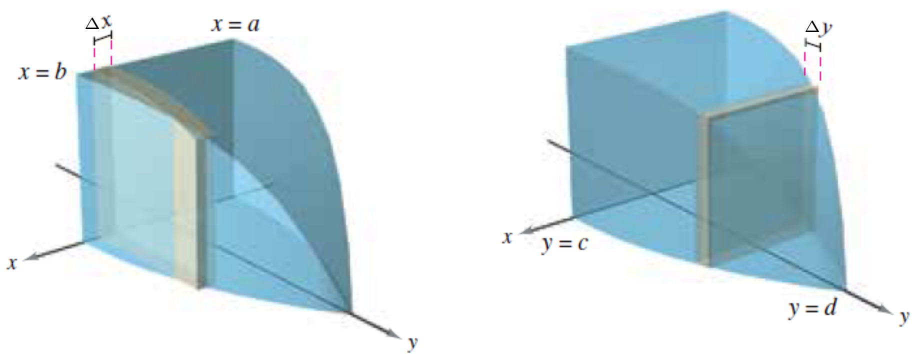
$$\pi \int_{-4}^{4} \left[(\sqrt{25-x^2}-0)^2 - (3-0)^2\right] dx = \frac{256\pi}{3}\,.$$

In general, the disk method can be used to compute a solid whose area of cross sections along a particular axis is known. Let D be a solid lies between two planes $x = a$ and $x = b$ $(a < b)$, and the area of the cross section of D taken perpendicular to the x-axis is $A(x)$, then

$$\text{the volume of D} = \int_a^b A(x)\,dx\,.$$

Similarly, if D lies between $y = c$ and $y = d$ $(c < d)$, and the area of the cross section of D taken perpendicular to the y-axis is $A(y)$, then

$$\text{the volume of D} = \int_c^d A(y)\,dy\,.$$

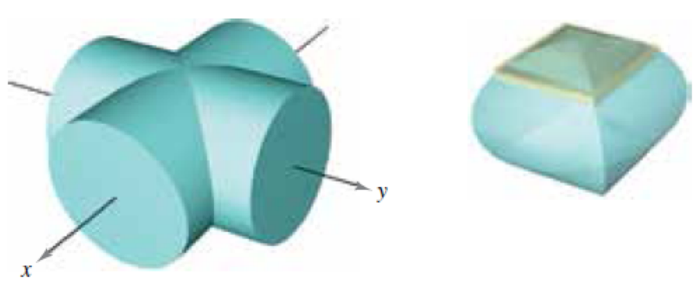


(a) Cross sections perpendicular to x-axis **(b)** Cross sections perpendicular to y-axis

Example 7.8. The volume of a cone with height h and base area A is given by

$$\int_0^h \frac{A(h-y)^2}{h^2}\,dy = -\frac{A}{h^2}\frac{1}{3}(h-y)^3\Big|_{y=0}^{y=h} = \frac{1}{3}Ah\,.$$

Example 7.9. Find the volume of the solid of intersection of the two right circular cylinders of radius r whose axes meet at right angles.



Two intersecting cylinders Solid of intersection

The area of cross sections taken perpendicular to the z-axis is given by

$$A(z) = (2\sqrt{r^2 - z^2})^2 = 4(r^2 - z^2)\,.$$

Therefore, the volume of the solid of intersection is given by

$$\int_{-r}^{r} 4(r^2 - z^2)\,dz = \frac{16}{3}r^3\,.$$