微積分 MA1001-A 上課筆記（精簡版）

2019.01.03.

Ching-hsiao Arthur Cheng 鄭經斅

8.5 Partial Fractions - 部份分式

In this section, we are concerned with the integrals $\int \frac{N(x)}{D(x)}\,dx$, where N, D are polynomial functions.

Write $N(x) = D(x)Q(x) + R(x)$, where Q, R are polynomials such that the degree of R is less than the degree of D (such an R is called a remainder). Then $\frac{N(x)}{D(x)} = R(x) + \frac{R(x)}{D(x)}$. Since it is easy to find the indefinite integral of R, it suffices to consider the case when the degree of the numerator is less than the degree of the denominator.

W.L.O.G., we assume that N and D no common factor, $\deg(N) < \deg(D)$, and the leading coefficient of D is 1. Since D is a polynomial with real coefficients,

$$D(x) = \Big(\prod_{j=1}^{m}(x+q_j)^{r_j}\Big)\Big(\prod_{j=1}^{n}(x^2+b_jx+c_j)^{d_j}\Big),$$

where $r_j, d_j \in \mathbb{N}$, $q_j \neq q_k$ for all $j \neq k$, $b_j \neq b_k$ or $c_j \neq c_k$ for all $j \neq k$, and $b_j^2 - 4c_j < 0$ for all $1 \leqslant j \leqslant m$. In other words, $-q_j$ are zeros of D with multiplicity r_j, and $\frac{-b_j \pm i\sqrt{4c_j - b_j^2}}{2}$ are zeros of D with multiplicity d_j, here $i = \sqrt{-1}$. Therefore,

$$\frac{N(x)}{D(x)} = \sum_{j=1}^{m}\Big[\sum_{\ell=1}^{r_j}\frac{A_{j\ell}}{(x+q_j)^{\ell}}\Big] + \sum_{j=1}^{n}\Big[\sum_{\ell=1}^{r_j}\frac{B_{j\ell}x + C_{j\ell}}{(x^2+b_j+c_j)^{\ell}}\Big] \tag{8.5.1}$$

for some constants $A_{j\ell}$, $B_{j\ell}$ and $C_{j\ell}$. Note that there are $\sum_{j=1}^{m} r_j + 2\sum_{j=1}^{n} d_j \equiv \deg(D)$ constants to be determined, and this can be done by the comparison of coefficients after the reduction of common denominator.

Example 8.24. Write $\frac{5x^2+20x+6}{x^3+2x^2+x}$ in the form of (8.5.1).

Note that $x^3+2x^2+x = x(x^2+2x+1) = x(x+1)^2$; thus to write the rational function above in the form of (8.5.1), we must have

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

for some constant A, B, C.

Multiplying both sides of the equality above by $x(x+1)^2$, we find that

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx = (A+B)x^2 + (2A+B+C)x + A\,;$$

thus A, B, C satisfy

$$\begin{aligned} A + B &= 5 \\ 2A + B + C &= 20 \\ A &= 6\,. \end{aligned}$$

Therefore, $A = 6$, $B = -1$ and $C = 9$; thus

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\,.$$

Example 8.25. Write $\dfrac{1}{x^4+1}$ in the form of (8.5.1).

Note that $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$, so

$$\frac{1}{x^4+1} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}\,.$$

Multiplying both sides of the equality above by $x^4 + 1$, we have

$$\begin{aligned} 1 &= (Ax + B)(x^2 - \sqrt{2}x + 1) + (Cx + D)(x^2 + \sqrt{2}x + 1) \\ &= (A + C)x^3 + (-\sqrt{2}A + B + \sqrt{2}C + D)x^2 + (A - \sqrt{2}B + C + \sqrt{2}D)x + (B + D)\,; \end{aligned}$$

thus comparing the coefficients, we find that A, B, C, D satisfy

$$\begin{aligned} A + C &= 0 \\ -\sqrt{2}A + B + \sqrt{2}C + D &= 0 \\ A - \sqrt{2}B + C + \sqrt{2}D &= 0 \\ B + D &= 1\,. \end{aligned}$$

Therefore, the first and the third equations imply that $A = -C$ and $B = D$; thus the second and the fourth equation shows that $A = -C = \dfrac{1}{2\sqrt{2}}$ and $B = D = \dfrac{1}{2}$. As a consequence,

$$\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\Big[\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} + \frac{-x + \sqrt{2}}{x^2 - \sqrt{2}x + 1}\Big]\,.$$

In order to find the integral of $\dfrac{N(x)}{D(x)}$, by writing $\dfrac{N(x)}{D(x)}$ in the form of (8.5.1), it suffices to find the integral of $\dfrac{B_{j\ell}x + C_{j\ell}}{(x^2 + b_j x + c_j)^\ell}$ for

$$\int \frac{A_{j\ell}}{(x + q_j)^\ell}\,dx = \begin{cases} \dfrac{A_{j\ell}}{1-\ell}(x + q_j)^{1-\ell} + C & \text{if } \ell \neq 1\,, \\ A_{j\ell} \ln|x + q_j| + C & \text{if } \ell = 1\,. \end{cases}$$

Note that

$$\frac{B_{j\ell}x + C_{j\ell}}{(x^2+b_jx+c)^\ell} = \frac{B_{j\ell}}{2}\frac{2x+b_j}{(x^2+b_jx+c_j)^\ell} + \big(C_{j\ell} - \frac{b_jB_{j\ell}}{2}\big)\frac{1}{(x^2+b_jx+c_j)^\ell}$$

and

$$\int \frac{2x+b_j}{(x^2+b_jx+c_j)^\ell}\,dx = \begin{cases} \dfrac{1}{1-\ell}(x^2+b_jx+c_j)^{1-\ell} + C & \text{if } \ell \neq 1\,, \\ \ln(x^2+b_jx+c_j) + C & \text{if } \ell = 1\,; \end{cases}$$

thus to find the integral of $\dfrac{B_{j\ell}x + C_{j\ell}}{(x^2+b_jx+c_j)^\ell}$, it suffices to compute $\displaystyle\int \frac{1}{(x^2+b_jx+c_j)^\ell}\,dx$.

Nevertheless, with a denoting the number $\dfrac{4c_j - b_j^2}{4}$,

$$\int \frac{1}{(x^2+b_jx+c_j)^\ell}\,dx = \int \frac{1}{\big[\big(x-\frac{b_j}{2}\big)^2 + \frac{4c_j-b_j^2}{4}\big]^\ell}\,dx = \int \frac{1}{\big[\big(x-\frac{b_j}{2}\big)^2 + a^2\big]^\ell}\,d\big(x-\frac{b_j}{2}\big)$$

which can be computed through the substitution $x - \dfrac{b_j}{2} = a\tan u$:

$$\int \frac{1}{\big[\big(x-\frac{b_j}{2}\big)^2 + a^2\big]^\ell}\,d\big(x-\frac{b_j}{2}\big) = a^{1-2\ell}\int \cos^{2\ell-2}u\,du\,.$$

Example 8.26. Find the indefinite integral $\displaystyle\int \frac{dx}{x^4+1}$.

Using the conclusion from Example 8.25, we find that

$$\begin{aligned}
\int \frac{dx}{x^4+1} &= \frac{1}{2\sqrt{2}}\int \Big[\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\Big]\,dx \\
&= \frac{1}{2\sqrt{2}}\int \Big[\frac{1}{2}\cdot\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{1}{2}\cdot\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\Big]\,dx \\
&\quad + \frac{1}{2\sqrt{2}}\int \Big[\frac{1}{2}\cdot\frac{\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{1}{2}\cdot\frac{\sqrt{2}}{x^2-\sqrt{2}x+1}\Big]\,dx \\
&= \frac{1}{4\sqrt{2}}\int \Big[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}}{\big(x+\frac{1}{\sqrt{2}}\big)^2 + (\frac{1}{\sqrt{2}})^2} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{\big(x-\frac{1}{\sqrt{2}}\big)^2 + (\frac{1}{\sqrt{2}})^2}\Big]\,dx \\
&= \frac{1}{4\sqrt{2}}\Big[\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + 2\arctan(\sqrt{2}x+1) + 2\arctan(\sqrt{2}x-1)\Big] + C\,.
\end{aligned}$$

Example 8.27. Find the indefinite integral $\displaystyle\int \frac{\sec x}{\tan^3 x}\,dx$.

Let $u = \sec x$. Then $du = \sec x\tan x$; thus

$$\int \frac{\sec x}{\tan^3 x}\,dx = \int \frac{\sec x\tan x}{\tan^4 x}\,dx = \int \frac{du}{(u^2-1)^2} = \int \frac{du}{(u+1)^2(u-1)^2}\,.$$

Write $\dfrac{1}{(u+1)^2(u-1)^2}$ is the form of (8.5.1):

$$\frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}\,,$$

where A, B, C, D satisfy

$$A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2 = 1\,.$$

Therefore, A, B, C, D satisfy

$$\begin{aligned} A + C &= 0 \\ -A + B + C + D &= 0 \\ -A - 2B - C + 2D &= 0 \\ A + B - C + D &= 1 \end{aligned}$$

which implies that $A = B = -C = D = \dfrac{1}{4}$. As a consequence,

$$\begin{aligned} \int \frac{du}{(u+1)^2(u-1)^2} &= \frac{1}{4}\int \Big[\frac{1}{u+1} + \frac{1}{(u+1)^2} - \frac{1}{u-1} + \frac{1}{u-1)^2}\Big]\,du \\ &= \frac{1}{4}\Big[\ln|u+1| - \frac{1}{u+1} - \ln|u-1| - \frac{1}{u-1}\Big] + C \\ &= \frac{1}{4}\Big[\ln\Big|\frac{u+1}{u-1}\Big| - \frac{2u}{u^2-1}\Big] + C\,; \end{aligned}$$

thus

$$\int \frac{\sec x}{\tan^3 x}\,dx = \frac{1}{4}\Big[\ln\Big|\frac{\sec x+1}{\sec x-1}\Big| - \frac{2\sec x}{\tan^2 x}\Big] + C\,.$$

Example 8.28. Find the indefinite integral $\displaystyle\int \frac{dx}{(1+x^n)^{\frac{1}{n}}}$, where n is a positive integer.

Let $1 + x^{-n} = u^n$. Then $x^n = \dfrac{1}{u^n-1}$ and $-x^{-n-1}\,dx = u^{n-1}\,du$; thus

$$\int \frac{dx}{(1+x^n)^{\frac{1}{n}}} = \int \frac{dx}{x(1+x^{-n})^{\frac{1}{n}}} = \int \frac{-x^n}{(1+x^{-n})^{\frac{1}{n}}}(-x^{-n-1})\,dx = -\int \frac{u^{n-2}}{u^n-1}\,du$$

which is the indefinite integral of a rational function of u and we know how to compute it. In particular, when $n = 4$,

$$\frac{u^2}{u^4-1} = \frac{u^2}{(u-1)(u+1)(u^2+1)} = \frac{1}{4}\cdot\frac{1}{u-1} - \frac{1}{4}\cdot\frac{1}{u+1} + \frac{1}{2}\cdot\frac{1}{u^2+1}\,;$$

thus

$$\int \frac{u^2}{u^4-1}\,du = \frac{1}{4}\ln|u-1| - \frac{1}{4}\ln|u+1| + \frac{1}{2}\arctan u + C$$

which further implies that

$$\int \frac{dx}{(1+x^4)^{\frac{1}{4}}} = \frac{1}{4}\ln\left|\frac{(1+x^{-4})^{\frac{1}{4}}-1}{(1+x^{-4})^{\frac{1}{4}}+1}\right| + \frac{1}{2}\arctan\left[(1+x^{-4})^{\frac{1}{4}}\right] + C\,.$$

- **The substitution of** $t = \tan\frac{x}{2}$

In Section 5.3 we have introduced the substitution $t = \tan\frac{x}{2}$ to find the anti-derivative of trigonometric functions. We recall that if $t = \tan\frac{x}{2}$, then

$$\sin x = \frac{2t}{1+t^2}\,, \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \text{and} \qquad dx = \frac{2dt}{1+t^2}\,.$$

Using this substitution, the anti-derivative of rational functions of sine and cosine can be computed via the integration of rational functions.

Example 8.29. Find the indefinite integral $\displaystyle\int \frac{\sec x}{\tan^3 x}\,dx$.

Rewriting the integrand, we have

$$\int \frac{\sec x}{\tan^3 x}\,dx = \int \frac{\cos^2 x}{\sin^3 x}\,dx\,.$$

Let $t = \tan\frac{x}{2}$. Then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$; thus

$$\begin{aligned}\int \frac{\sec x}{\tan^3 x}\,dx &= \int \frac{\frac{(1-t^2)^2}{(1+t^2)^2}}{\frac{(2t)^3}{(1+t^2)^3}}\frac{2dt}{1+t^2} = \frac{1}{4}\int \frac{(1-t^2)^2}{t^3}\,dt = \frac{1}{4}\int \left(t^{-3}-2t^{-1}+t\right)dt\\ &= \frac{1}{4}\left[-\frac{1}{2}t^{-2} - 2\ln|t| + \frac{1}{2}t^2\right] + C\\ &= \frac{1}{8}\left[\tan^2\frac{x}{2} - \cot^2\frac{x}{2}\right] - \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C\,.\end{aligned}$$

Example 8.30. Find the indefinite integral $\displaystyle\int \frac{1}{2+\sin x}\,dx$.

Let $t = \tan\frac{x}{2}$. Then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$; thus

$$\begin{aligned}\int \frac{1}{2+\sin x}\,dx = \int \frac{1}{2+\frac{2t}{1+t^2}}\frac{2dt}{1+t^2} = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ = \frac{2}{\sqrt{3}}\arctan\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C = \frac{2}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}} + C \\ = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\tan\frac{x}{2} + \frac{1}{\sqrt{3}}\right) + C\,.\end{aligned}$$