Extra Exercise Problem Sets 5

Nov. 21. 2018

Problem 1. 1. Show that for any fixed constant c, if $x > c$ then there exists $d \in [c, x]$ such that

$$\frac{\ln x}{x} = \frac{c}{x} + \frac{x-c}{dx}\,. \tag{$\star$}$$

Hint: Write $\ln x = \ln c + \int_c^x \frac{1}{t}\,dt$ and use the mean value theorem for integrals.

2. Conclude from ($\star$) that $\lim_{x\to\infty} \frac{\ln x}{x} = 0$. An ε-M argument will be welcome.

3. Show that there is no asymptote for the graph of $y = \ln x$.

Problem 2. Show the following two inequalities.

$$\ln(1+x) \geqslant \sum_{k=1}^{2\ell} \frac{(-1)^{k-1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \frac{x^{2\ell}}{2\ell} \qquad \text{if } x \in (0,1)\,,$$

$$\ln(1+x) \leqslant \sum_{k=1}^{2\ell-1} \frac{(-1)^{k-1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^{2\ell-1}}{2\ell-1} \qquad \text{if } x \in (0,1)\,.$$

Hint: Follow the example we did in class and you will need that for $|x| < 1$, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=0}^{\infty} (-x)^k$.