Extra Exercise Problem Sets 4

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Problem 1. Let $f : [a, b] \to \mathbb{R}$ be a function, and f is Riemann integrable on $[a, b]$. Show that f must be bounded on $[a, b]$; that is, there exists a real number $M > 0$ such that $|f(x)| \leqslant M$ for all $a \leqslant x \leqslant b$.

Problem 2. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b \cos x \, dx$ by the following steps.

(a) Partition $[a, b]$ into n sub-intervals with equal length. Write down the Riemann sum using the right end-point rule.

(b) Prove that

$$\sum_{i=1}^{n} \cos(a + id) = \frac{\sin\left[a + \left(n + \frac{1}{2}\right)d\right] - \sin\left(a + \frac{d}{2}\right)}{2 \sin \frac{d}{2}} . \qquad (\star)$$

Hint: Use the sum and difference formula $\sin(\vartheta + \varphi) - \sin(\vartheta - \varphi) = 2 \sin \vartheta \cos \varphi$.

(c) Use $(\star)$ to simplify the Riemann sum in (a), and find the limit of the Riemann sum as n approaches infinity. Show that

$$\int_a^b \cos x \, dx = \sin b - \sin a \, .$$

Problem 3. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b x^m \, dx$, where m is a non-negative integer, by the following steps.

(a) Partition $[a, b]$ into n sub-intervals with equal length. Show that the Riemann sum using the right end-point rule is given by

$$I_n = \sum_{k=0}^{m} \left[C_k^m a^{m-k} (b-a)^{k+1} \left(\frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k \right) \right] ,$$

where $C_k^m = \dfrac{m!}{k!(m-k)!}$.

(b) Show that

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1}(n+1)^{k+1} - \frac{1}{k+1}\left[C_{k-1}^{k+1} \sum_{i=1}^{n} i^{k-1} + \cdots + C_1^{k+1} \sum_{i=1}^{n} i + (n+1) \right] . \qquad (\star\star)$$

Hint: Expand $(j+1)^k$ for $j = 0, 1, 2, \cdots, n$ by the binomial expansion formula, and sum over j to obtain the equality above.

(c) Use $(\star\star)$ to show that $\displaystyle\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k = \frac{1}{k+1}$.

(d) Use the limit in (c) to find the limit of the Riemann sum in (a) by passing to the limit as n approaches infinity. Simplify the result to show that

$$\int_a^b x^m \, dx = \frac{b^{m+1} - a^{m+1}}{m+1} \,.$$

Problem 4. Let $a > 0$ and $b > 1$. Compute $\int_1^b \log_a x \, dx$ by the following steps.

(a) Partition $[1, a]$ into n sub-intervals by $x_i = r^i$, where $1 \leqslant i \leqslant n$ and $r = b^{\frac{1}{n}}$. Show that the Riemann sum given by the right end-point rule is

$$(r-1) \log_a r \sum_{i=1}^{n} i r^{i-1} \,. \qquad (\diamond)$$

(b) Use the fact that $\frac{d}{dr} r^i = i r^{i-1}$ to find the sum of $i r^{i-1}$ and show that

$$\sum_{i=1}^{n} i r^{i-1} = \frac{n r^{n+1} - (n+1) r^n + 1}{(r-1)^2} \,. \qquad (\diamond\diamond)$$

(c) Use $(\diamond)$ and $(\diamond\diamond)$ to simplify the Riemann sum given by the right end-point rule and show that the Riemann sum is

$$\frac{nbr - nb - b + 1}{n(r-1)} \log_a b = \left[b - \frac{b-1}{n(r-1)} \right] \log_a b \,.$$

(d) Assuming that you know $\frac{d}{dx}\Big|_{x=0} b^x = A(a) \log_a b$ for some constant $A > 0$ depending on a, show that

$$\int_1^b \log_a x \, dx = b \log_a b - \frac{b-1}{A(a)} \,.$$

Problem 5. Determine the following limits by identifying the limits as limits of certain Riemann sums so that the limits are the same as certain integrals.

1. $\displaystyle \lim_{n \to \infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{n^{\frac{3}{2}}}$.

2. $\displaystyle \lim_{n \to \infty} \left[\frac{1}{\sqrt{n^2 + 2n}} + \frac{1}{\sqrt{n^2 + 4n}} + \frac{1}{\sqrt{n^2 + 6n}} + \cdots + \frac{1}{\sqrt{n^2 + 2n^2}} \right]$.