Calculus MA1001-A Sample Midterm 1

National Central University, Oct. 29, 2018

Problem 1. 定義題或定理敘述題共兩小題！

Problem 2. Compute the following limits (without using L'Hôpital's rule in Problem 3).

(1) $\lim_{x \to 0} \frac{\sin(x^2)}{|x|}$ (2) $\lim_{x \to 0} \frac{\sqrt[3]{1+x^2}-1}{\sin(x^2)}$.

Problem 3. Let $f : (-\pi, \pi) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \sin(x^2)\cos(\cot x) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$ Find the derivatives of f.

Problem 4. Complete the following.

(1) Show the Cauchy mean value theorem: Let $f, g : [a, b] \to \mathbb{R}$ be continuous and f, g are differentiable on (a, b). Show that if $g(a) \neq g(b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}\,.$$

(2) Show that if $f : [a, b] \to \mathbb{R}$ is continuous and f is twice differentiable on (a, b) (that is, $f''(x)$ exists for all $x \in (a, b)$), then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2\,.$$

(3) Show that L'Hôpital's rule: Let $f, g : (a, b) \to \mathbb{R}$ be differentiable, $c \in (a, b)$, and $f(c) = g(c) = 0$. Suppose that $\frac{f(x)}{g(x)}$ and $\frac{f'(x)}{g'(x)}$ are both defined on (a, b), except possibly at c. Show that if the limit $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}\,.$$

(4) Use L'Hôpital's rule to compute the the limit in Problem 1.

Problem 5. Suppose that $f : (0, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to (0, \infty)$ are two strictly increasing, differentiable functions satisfying

$$f(g(x)) = x \quad \forall\, x \in \mathbb{R}\,, \qquad g(f(x)) = x \quad \forall\, x \in (0, \infty)\,,$$

and $f(ab) = f(a) + f(b)$ for all $a, b > 0$. Show that $xf'(x)g'(x) = g(x)$ for all $x > 0$.

Hint: Differentiate the relation $f(cx) = f(x) + f(c)$ and then let $c = \frac{g(x)}{x}$.

Problem 6. Suppose that x and y satisfy the relation $1 + x = \sin(x + y^2)$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-1, 1)$ using the implicit differentiation.

Problem 7. Let $f : [0, 2\pi] \to \mathbb{R}$ be given by $f(x) = -2\cos x - \frac{1}{2}\sin(2x)$.

(1) Find the inflection points of the graph of f.

(2) Use the second derivative test to find all the relative extrema of f'.

(3) Show that $|f(x) - f(y)| \leqslant 3|x - y|$ for all $x, y \in [0, 2\pi]$.