Calculus MA1001-A Quiz 04

National Central University, Oct. 11 2018

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Problem 1. (4pts) State the intermediate value theorem.

Problem 2. (3pts) Suppose that you know that the function $y = 2^x$ is continuous on $\mathbb{R}$, $2^a > 2^b$ for all $a > b$, and the two limits $\lim\limits_{x\to\infty} 2^x = \infty$, $\lim\limits_{x\to-\infty} 2^x = 0$. Find all the asymptotes of the graph of the function

$$f(x) = \frac{2^x + 2^{-x}}{2^x - 2^{-x}}\,.$$

Solution: First we find the vertical asymptote. Solving for $2^x - 2^{-x} = 0$, we find that $x = 0$; thus $x = 0$ is the only vertical asymptote.

For the horizontal asymptote, we note that if $x \neq 0$,

$$f(x) = \frac{1 + 2^{-2x}}{1 - 2^{-2x}} = \frac{1 + (2^{-x})^2}{1 - (2^{-x})^2} \quad \text{and} \quad f(x) = \frac{2^{2x} + 1}{2^{2x} - 1} = \frac{(2^x)^2 + 1}{(2^x)^2 - 1}\,.$$

Since $\lim\limits_{x\to\infty} \big[1 + (2^{-x})^2\big] = 1$; $\lim\limits_{x\to\infty} \big[1 - (2^{-x})^2\big] = 1$; $\lim\limits_{x\to-\infty} \big[(2^x)^2 + 1\big] = 1$ and $\lim\limits_{x\to-\infty} \big[(2^x)^2 - 1\big] = -1$, we find that

$$\lim_{x\to\infty} f(x) = \frac{\lim\limits_{x\to\infty} \big[1 + (2^{-x})^2\big]}{\lim\limits_{x\to\infty} \big[1 - (2^{-x})^2\big]} = \frac{1}{1} = 1$$

and

$$\lim_{x\to-\infty} f(x) = \frac{\lim\limits_{x\to-\infty} \big[(2^x)^2 + 1\big]}{\lim\limits_{x\to\infty} \big[(2^x)^2 - 1\big]} = \frac{1}{-1} = -1\,.$$

Therefore, there are two horizontal asymptotes: $y = \pm 1$.

Problem 3. (3pts) Find the tangent lines of the graph of the function $y = x^2$ that passes through the point $(1, -3)$.

Solution: Suppose that a tangent line passes through point (a, a^2). Then the slope of this tangent line is $2a$ so that the tangent line is

$$y = 2a(x - a) + a^2\,.$$

Since this line passes through $(1, -3)$, we must have $-3 = 2a(1 - a) + a^2$; thus $a^2 - 2a - 3 = 0$ which shows $a = 3$ or $a = -1$. Therefore, the tangent lines passing through $(1, -3)$ are

$$y = 6(x - 3) + 9 = 6x - 9 \quad \text{and} \quad y = -2(x + 1) + 1 = -2x - 1\,.$$