Calculus II Midterm 3

National Central University, Summer Session 2012, Aug. 28, 2012

Problem 1. (20%) Find the local maximum and minimum values and saddle points of the function $f(x, y) = e^y(y^2 - x^2)$.

Sol: Since $f_x(x, y) = -2xe^y$ and $f_y(x, y) = e^y(y^2 - x^2) + 2ye^y = e^y(y^2 + 2y - x^2)$, the critical points of f are $(0, 0)$ and $(0, -2)$. Moreover, since

$$f_{xx}(x, y) = -2e^y, \qquad f_{xy}(x, y) = f_{yx}(x, y) = -2xe^y,$$
$$f_{yy}(x, y) = e^y(y^2 + 2y - x^2) + e^y(2y + 2) = e^y(y^2 + 4y + 2 - x^2),$$

we have

$$f_{xx}(0, 0) = -2, \qquad f_{xy}(0, 0) = f_{yx}(0, 0) = 0, \qquad f_{yy}(0, 0) = 2;$$
$$f_{xx}(0, -2) = -2e^{-2}, \quad f_{xy}(0, -2) = f_{yx}(0, -2) = 0, \quad f_{yy}(0, -2) = -2e^{-2}.$$

Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2$,

1. Since $D(0, 0) < 0$, $(0, 0)$ is a saddle point.
2. Since $D(0, -2) = 4e^{-4} > 0$ and $f_{xx}(0, -2) < 0$, f attains its local maximum $f(0, -2) = 4e^{-2}$ at $(0, -2)$.
3. Since there is no other critical point, there is no local minimum of f. □

Problem 2. (20%) Find the extreme values of $f(x, y, z) = x+2y$ subject to the constraints $x+y+z = 1$ and $y^2 + z^2 = 4$.

Sol: Let $g(x, y, z) = x + y + z - 1$ and $h(x, y, z) = y^2 + z^2 - 4$. Suppose that f attains its extreme value (subject to the constraints) at (x, y, z). Then there exist two constants λ and μ such that

$$(\nabla f)(x, y, z) = \lambda(\nabla g)(x, y, z) + \mu(\nabla h)(x, y, z),$$
$$g(x, y, z) = 0, \qquad h(x, y, z) = 0$$

or equivalently,

$$1 = \lambda, \tag{0.1a}$$
$$2 = \lambda + 2\mu y, \tag{0.1b}$$
$$0 = \lambda + 2\mu z, \tag{0.1c}$$
$$x + y + z = 1, \tag{0.1d}$$
$$y^2 + z^2 = 4. \tag{0.1e}$$

Using (0.1a) in (0.1b,c), we find that

$$2\mu y = -2\mu z = 1$$

which implies that $y = -z$. Therefore, (0.1e) suggests that $y = \pm\sqrt{2}$ and $z = \mp\sqrt{2}$; thus $x = 1$.

1. For $(x,y,z)=(1,\sqrt{2},-\sqrt{2})$, $f(1,\sqrt{2},-\sqrt{2})=1+2\sqrt{2}$.

2. For $(x,y,z)=(1,-\sqrt{2},\sqrt{2})$, $f(1,-\sqrt{2},\sqrt{2})=1-2\sqrt{2}$.

Therefore, the maximum of f subject to $g=h=0$ is $1+2\sqrt{2}$, and the minimum is $1-2\sqrt{2}$. □

Problem 3. Suppose that the double integral $\iint_D 3x^2 dA$ can be computed by the iterated integral $\int_1^2\int_0^{\ln x} 3x^2 dydx$. Complete the following.

1. (10%) Directly evaluate the iterated integral.

2. (10%) Sketch the region of integration D.

3. (10%) Evaluate the double integral by reversing the order of integration.

Sol:

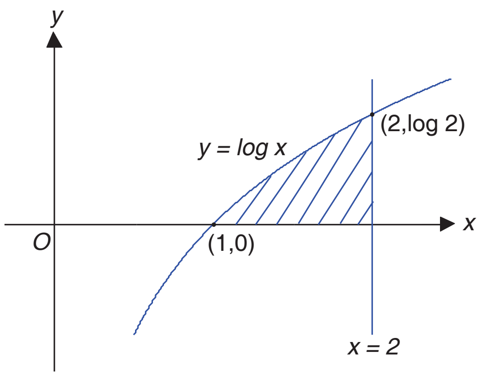
1. Integrating in y first:

$$\int_1^2\int_0^{\ln x} 3x^2 dydx = \int_1^2 3x^2 y\Big|_{y=0}^{y=\ln x} dx = \int_1^2 3x^2\ln x dx\,.$$

Let $u=\ln x$ and $dv=3x^2$. Then $du=\frac{1}{x}dx$ and $v=x^3$. Integrating by parts,

$$\int_1^2 3x^2\ln x dx = x^3\ln x\Big|_{x=1}^{x=2} - \int_1^2 x^3\frac{1}{x}dx = 8\ln 2 - \frac{1}{3}x^3\Big|_{x=1}^{x=2} = 8\ln 2 - \frac{7}{3}\,.$$

2. Since $1\le x\le 2$, $0\le y\le \ln x$, the region is



3. $y=\ln x$ if and only if $x=e^y$. Therefore,

$$\int_1^2\int_0^{\ln x} 3x^2 dydx = \int_0^{\ln 2} x^3\Big|_{x=e^y}^{x=2} dy = \int_0^{\ln 2} 8-e^{3y}dy = (8y-\frac{1}{3}e^{3y})\Big|_{y=0}^{y=\ln 2}$$
$$= 8\ln 2 - \frac{1}{3}(e^{3\ln 2}-1) = 8\ln 2 - \frac{7}{3}\,.$$

Problem 4. (20%) Evaluate the double integral $\iint_D \arctan \frac{y}{x}\, dA$ using the polar coordinate, where

$$D = \{(x, y) \mid 1 \le x^2 + y^2 \le 4, 0 \le y \le x\}.$$

Sol: The region D in polar coordinate can be written as $D = \{(r, \theta) | 1 \le r \le 2, 0 \le \theta \le \frac{\pi}{4}\}$. Therefore,

$$\iint_D \arctan \frac{y}{x}\, dA = \int_0^{\frac{\pi}{4}} \int_1^2 \arctan \frac{r\sin\theta}{r\cos\theta}\, r dr d\theta = \int_0^{\frac{\pi}{4}} \int_1^2 r\theta dr d\theta = \Big(\int_0^{\frac{\pi}{4}} \theta d\theta\Big)\Big(\int_1^2 r dr\Big) = \frac{3\pi^2}{64}. \quad \square$$

Problem 5. (20%) The boundary of a lamina consists of the semicircles $y = \sqrt{1 - x^2}$ and $y = \sqrt{4 - x^2}$ together with the portions of the x-axis that join them. Find the center of mass of the lamina if the density ρ at (x, y) is given by $\rho(x, y) = \sqrt{x^2 + y^2}$.

Sol: Let $D = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$. Then the mass M is

$$M = \iint_D \rho(x, y) dA = \int_0^\pi \int_1^2 r^2 dr d\theta = \frac{7\pi}{3},$$

and the moment about the x and y-axis are

$$M_x = \iint_D y\rho(x, y) dA = \int_0^\pi \int_1^2 r^3 \sin\theta dr d\theta = \frac{15}{2},$$
$$M_y = \iint_D x\rho(x, y) dA = \int_0^\pi \int_1^2 r^3 \cos\theta dr d\theta = 0.$$

Therefore, the center of mass is $(\frac{M_y}{M}, \frac{M_x}{M}) = (0, \frac{45}{14\pi})$. $\square$

Problem 6. Let D be the intersection of two solid cylinders $x^2 + y^2 \le 1$ and $x^2 + z^2 \le 1$.

1. (20%) Find the volume of D.
2. (20%) Find the surface area of the boundary of D.

Sol: Let $z = f(x, y) = \sqrt{1 - x^2}$, and $R = \{(x, y) \mid x^2 + y^2 \le 1\}$.

1. The volume of D is

$$2\iint_R f(x, y) dA = 2\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2} dy dx = 2\int_{-1}^1 \Big[\sqrt{1 - x^2}\, y\Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\Big] dx$$
$$= 4\int_{-1}^1 (1 - x^2) dx = 4\Big[x - \frac{1}{3}x^3\Big]\Big|_{x=-1}^{x=1} = \frac{16}{3}.$$

2. Since $f_x(x, y) = -\frac{x}{\sqrt{1 - x^2}}$ and $f_y(x, y) = 0$, we have

$$\sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} = \sqrt{1 + \frac{x^2}{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}};$$

thus the surface area of the boundary of D is

$$4\iint_R \frac{1}{\sqrt{1 - x^2}} dA = 4\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1 - x^2}} dy dx = 4\int_{-1}^1 \Big[\frac{y}{\sqrt{1 - x^2}}\Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\Big] dx$$
$$= 4\int_{-1}^1 2 dx = 16.$$

$\square$