Calculus II Midterm 1

National Central University, Summer Session 2012, Aug. 14, 2012

Problem 1. Let C_1 be the polar graph of the polar function $r = 1 + \cos\theta$, and C_2 be the polar graph of the polar function $r = 3\cos\theta$ (see figure 1).

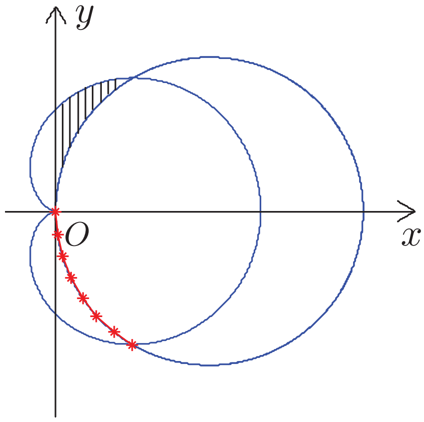

Figure 1

1. (10%) Find the intersection points of C_1 and C_2.
2. (10%) Find the line L passing through the lowest intersection point and tangent to the curve C_2.
3. (5%) Identify the curve marked by $\star$ on the θ-r plane for $0 \le \theta \le 2\pi$.
4. (10%) Find the area of the shaded region.

Sol:

1. Let $1 + \cos\theta = 3\cos\theta$. Then $2\cos\theta = 1$ or $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$. From the figure, it is also clear that C_1 and C_2 intersection at the origin. Therefore, the points of intersections are
$$\Big(\frac{3}{4}, \frac{3\sqrt{3}}{4}\Big), \quad \Big(\frac{3}{4}, -\frac{3\sqrt{3}}{4}\Big), \quad (0,0).$$

2. C_2 can be parametrized by $\big\{(x,y) \in \mathbb{R}^2 \,\big|\, x = 3\cos^2\theta, y = 3\cos\theta\sin\theta\big\}$. Therefore,
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3\sin^2\theta + 3\cos^2\theta}{-6\cos\theta\sin\theta} = \frac{\sin^2\theta - \cos^2\theta}{2\cos\theta\sin\theta};$$
thus at the lowest point of intersection $\big(\theta = \frac{5\pi}{3}\big)$, $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{3}}$. As a consequence, the desired tangent line is
$$y = -\frac{1}{\sqrt{3}}\Big(x - \frac{3}{4}\Big) - \frac{3\sqrt{3}}{4} = -\frac{x}{\sqrt{3}} - \frac{\sqrt{3}}{2}.$$

3. The curve marked by $\star$ is in the fourth quadrant, on the circle $r = 3\cos\theta$ with end-points $(0,0)$ and $\left(\frac{3}{4}, -\frac{3\sqrt{3}}{4}\right)$. Therefore, it corresponds to the curves marked by $\star$ is shown in Figure 2.

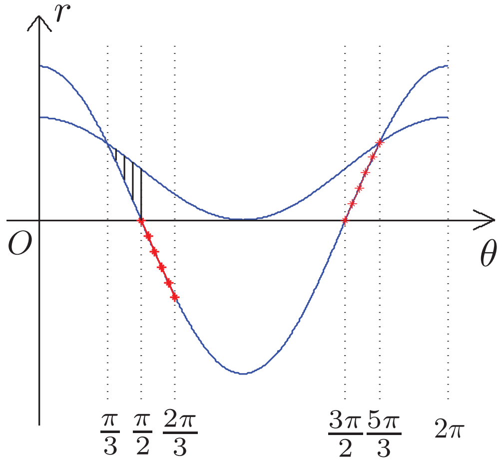

Figure 2

4. The shaded region on xy-plane corresponds to the shaded region in Figure 2. Therefore, the area of the shaded region (on xy-plane) is

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \Big[(1+\cos\theta)^2 - 9\cos^2\theta\Big]\,d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \Big[1 + 2\cos\theta - 4(1+\cos 2\theta)\Big]\,d\theta$$
$$= \Big[-3\theta + 2\sin\theta - 2\sin 2\theta\Big]\Big|_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}} = 2 - \frac{\pi}{2}.$$

Problem 2. (15%) Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\right\}_{n=1}^{\infty}$ is a decreasing sequence.

Proof. Let $f(x) = \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}$. Then $f(x) = \exp\left(\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)\right)$; thus

$$\begin{aligned} f'(x) &= \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}\left[\ln\left(1+\frac{1}{x}\right) + \left(x+\frac{1}{2}\right)\frac{1}{1+\frac{1}{x}}\frac{-1}{x^2}\right] \\ &= \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}\left[\ln\left(1+\frac{1}{x}\right) - \left(\frac{x}{2}+\frac{x+1}{2}\right)\frac{1}{x(x+1)}\right] \\ &= \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}\left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{2(x+1)} - \frac{1}{2x}\right]. \end{aligned}$$

It suffices to show that $g(x) \equiv \ln\left(1+\frac{1}{x}\right) - \frac{1}{2(x+1)} - \frac{1}{2x}$ is non-positive. Nevertheless,

$$\begin{aligned} g'(x) &= -\frac{1}{x(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{2x^2} \\ &= \frac{x^2 + (x+1)^2 - 2x(x+1)}{2x^2(x+1)^2} = \frac{1}{2x^2(x+1)^2} > 0; \end{aligned}$$

thus g is increasing. Moreover, $\lim\limits_{x\to\infty} g(x) = 0$; thus $g(x) < 0$ for all $x > 1$. $\square$

Problem 3.

1. (10%) Find all the values of p for which the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$ is convergent.

2. (10%) Find all the positive integers k for which the series $\sum_{n=1}^{\infty}\frac{(n!)^2}{(kn)!}$ is convergent.

Sol:

1. Since the series is an alternating series. It converges if $p_n = \frac{(\ln n)^p}{n}$ is decreasing and approaches 0 as $n \to \infty$. Nevertheless, if $p < k \in \mathbb{N}$, by L'Hospital's rule,

$$\lim_{x\to\infty}\frac{(\ln x)^p}{x} = \lim_{x\to\infty}\frac{p(\ln x)^{p-1}}{x} = p(p-1)\cdots(p-k+1)\lim_{x\to\infty}\frac{(\ln x)^{p-k}}{x} = 0.$$

   Moreover, let $f(x) = \frac{(\ln x)^p}{x}$. Then

$$f'(x) = \frac{p(\ln x)^{p-1} - (\ln x)^p}{x^2} = \frac{(\ln x)^{p-1}}{x^2}(p - \ln x)$$

   which implies that $f' < 0$ if $x \gg 1$. Therefore, the series is convergent for all $p \in \mathbb{R}$.

2. Let $a_n = \frac{(n!)^2}{(kn)!}$. Then $a_n \geq 0$.

   (a) The case $k = 1$. In this case $a_n \to \infty$ as $n \to \infty$ if $k = 1$. So the series is divergent if $k = 1$.

   (b) If $k \geq 2$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2(kn)!}{(k(n+1))!} \geq \frac{(n+1)^2(kn)!}{[k(n+1)][k(n+1)-1][k(n+1)-2](kn)!};$$

   thus $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0$. Therefore, the series is convergent (for $k \geq 2$) by the ratio test. □

Problem 4. (10%) Test the series $\sum_{n=2}^{\infty}(\sqrt[n]{2} - 1)$ for convergence or divergence.

Sol:

Method 1: Let $a_n = \sqrt[n]{2} - 1$. $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2^{1/n}-1}{\frac{1}{n}} = \lim_{n\to\infty}\frac{-2^{1/n}\ln 2\frac{1}{n^2}}{-\frac{1}{n^2}} = \ln 2.$$

Therefore, by the limit comparison test, $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=2}^{\infty} b_n$ converges or diverges together. On the other hand, the harmonic series $\sum_{n=2}^{\infty} b_n$ diverges; thus $\sum_{n=2}^{\infty}(\sqrt[n]{2} - 1)$ diverges.

Method 2: It is clear that the sequence $a_n = \sqrt[n]{2} - 1$ is decreasing to 0. By the integral test, the series is convergent if and only if the integral $\int_1^\infty (2^{1/x} - 1)dx$ is finite. Let $y = 2^{1/x} - 1$. Then $x = \dfrac{1}{\ln(y+1)}$; thus $dx = \dfrac{-dy}{(y+1)(\ln(y+1))^2}$

$$\int_1^\infty (2^{1/x} - 1)dx = \int_0^1 \frac{ydy}{(y+1)[\ln(y+1)]^2} \overset{(y+1=z)}{=} \int_1^2 \frac{(z-1)dz}{z(\ln z)^2} \overset{(z=e^t)}{=} \int_0^{\ln 2} \frac{e^t - 1}{t^2} dt.$$

Since $e^t \geq 1 + t$ if $t > 0$ (by the Taylor series),

$$\int_1^\infty (2^{1/x} - 1)dx \geq \int_0^{\ln 2} \frac{dt}{t} = \infty;$$

thus the series $\displaystyle\sum_{n=2}^\infty (\sqrt[n]{2} - 1)$ is divergent. □

Problem 5. (10%) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=2}^\infty \frac{x^{2n}}{n(\ln n)^2}.$$

Sol: Let $a_n = \dfrac{x^{2n}}{n(\ln n)^2}$. Then $a_n \geq 0$, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n(\ln n)^2 x^2}{(n+1)[\ln(n+1)]^2} = \left[\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right]^2 x^2 = x^2;$$

thus the series $\displaystyle\sum_{n=2}^\infty \frac{x^{2n}}{n(\ln n)^2}$ is convergent if $x^2 < 1$ and divergent if $x^2 > 1$. Therefore, the radius of convergence is $R = 1$.

As for the interval of convergence, we check if the series converges at $x = \pm 1$. Nevertheless, since

$$\int_2^\infty \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^\infty \frac{e^t}{e^t t^2} dt = \int_{\ln 2}^\infty \frac{1}{t^2}\, dt < \infty;$$

thus the series $\displaystyle\sum_{n=2}^\infty \frac{x^{2n}}{n(\ln n)^2}$ converges at $x = \pm 1$. Therefore, the interval of convergence if $[-1, 1]$. □

Problem 6. (10%) Let $f^{(k)}$ denote $\dfrac{d^k f}{dx^k}$, the k-th derivative of f, and $f^{(0)} \equiv f$. Suppose that $f^{(k)}$ is continuous for all $k \in \mathbb{N} \cup \{0\}$. Show that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + (-1)^n \int_a^x \frac{(t-x)^n}{n!} f^{(n+1)}(t)\, dt \qquad (0.1)$$

by the integration by parts formula and induction.

Proof. By the fundamental theorem of Calculus and integration by parts,

$$f(x) = f(a) + \int_a^x f'(t)dt = f(a) + (t-x)f'(t)\Big|_{t=a}^{t=x} - \int_a^x (t-x)f''(t)dt$$
$$= f(a) + f'(a)(x-a) - \int_a^x (t-x)f''(t)dt.$$

This prove the case $n = 1$.

Integrating by parts again suggests that

$$\int_a^x \frac{(t-x)^N}{N!} f^{(N+1)}(t)dt = \frac{(t-x)^{N+1}}{(N+1)!} f^{(N+1)}(t)\Big|_{t=a}^{t=x} - \int_a^x \frac{(t-x)^{N+1}}{(N+1)!} f^{(N+2)}(t)dt$$
$$= (-1)^{N+2} \frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1} - \int_a^x \frac{(t-x)^{N+1}}{(N+1)!} f^{(N+2)}(t)dt.$$

Now suppose that (0.1) holds for $n = N$. Then the identity above implies that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N + (-1)^N \int_a^x \frac{(t-x)^N}{N!} f^{(N+1)}(t)\, dt$$
$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1}$$
$$+ (-1)^{N+1} \int_a^x \frac{(t-x)^{N+1}}{(N+1)!} f^{(N+2)}(t)dt.$$

This implies that (0.1) also holds for $n = N + 1$. Therefore, (0.1) holds for all $n \in \mathbb{N} \cup \{0\}$ by induction. □