Calculus II Midterm 3

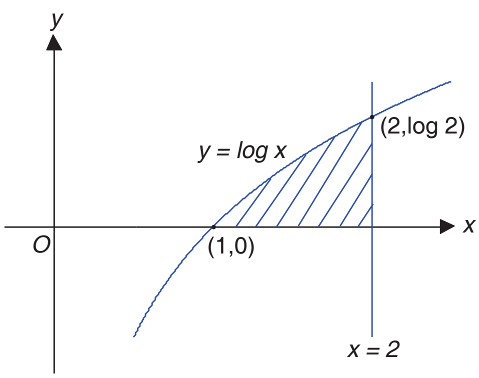
National Central University, Spring 2012, June 01, 2012

Problem 1. Evaluate the double integral $\int_1^2 \int_0^{\ln x} 3x^2 dydx$ in the following way:

1. (8%) Directly integrate by computing the iterated integral (You will need to integrate by parts to obtain the integral in x).
2. (4%) Sketch the region of integration.
3. (8%) Interchange the order of integration, and evaluate the double integral again.

Sol:

1. Integrating in y first:
$$\int_1^2 \int_0^{\ln x} 3x^2 dydx = \int_1^2 3x^2 y\Big|_{y=0}^{y=\ln x} dx = \int_1^2 3x^2 \ln x dx\,.$$
Let $u = \ln x$ and $dv = 3x^2$. Then $du = \frac{1}{x}dx$ and $v = x^3$. Integrating by parts,
$$\int_1^2 3x^2 \ln x dx = x^3 \ln x\Big|_{x=1}^{x=2} - \int_1^2 x^3 \frac{1}{x} dx = 8\ln 2 - \frac{1}{3}x^3\Big|_{x=1}^{x=2} = 8\ln 2 - \frac{7}{3}\,.$$
2. Since $1 \le x \le 2$, $0 \le y \le \ln x$, the region is



3. $y = \ln x$ if and only if $x = e^y$. Therefore,
$$\int_1^2 \int_0^{\ln x} 3x^2 dydx = \int_0^{\ln 2} x^3\Big|_{x=e^y}^{x=2} dy = \int_0^{\ln 2} 8 - e^{3y} dy = (8y - \frac{1}{3}e^{3y})\Big|_{y=0}^{y=\ln 2}$$
$$= 8\ln 2 - \frac{1}{3}(e^{3\ln 2} - 1) = 8\ln 2 - \frac{7}{3}\,.$$

Problem 2. Let D be the intersection of two solid cylinders $x^2 + y^2 \le 1$ and $x^2 + z^2 \le 1$.

1. (10%) Using the cylindrical coordinate to describe the region D. In other words, find the corresponding domain of D in the (r, θ, z) space (Suppose D is the same as $a \le r \le b$, $\alpha \le \theta \le \beta$, and $F_1(r,\theta) \le z \le F_2(r,\theta)$, find a, b, α, β as well as F_1, F_2).

2. (15%) Find the volume of D, or $\iiint_D dV$, using the cylindrical coordinate.

Sol:

1. We have $a = 0$, $b = 1$, $\alpha = 0$, $\beta = 2\pi$, and $F_1(r,\theta) = -\sqrt{1-r^2\cos^2\theta}$, $F_2(r,\theta) = \sqrt{1-r^2\cos^2\theta}$; that is,

$$D = \left\{(r,\theta,z) \,\middle|\, (r,\theta) \in [0,1]\times[0,2\pi], -\sqrt{1-r^2\cos^2\theta} \le z \le \sqrt{1-r^2\cos^2\theta}\right\}.$$

2. The volume of D can be computed by

$$\begin{aligned}
\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2\cos^2\theta}}^{\sqrt{1-r^2\cos^2\theta}} r\,dz\,dr\,d\theta &= \int_0^{\frac{\pi}{2}}\int_0^1 8r\sqrt{1-r^2\cos^2\theta}\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} -\frac{8}{3\cos^2\theta}\left(1-r^2\cos^2\theta\right)^{\frac{3}{2}}\Big|_{r=0}^{r=1} d\theta \\
&= -\frac{8}{3}\int_0^{\frac{\pi}{2}} \frac{\sin^3\theta - 1}{\cos^2\theta} d\theta = -\frac{8}{3}\int_0^{\frac{\pi}{2}} \frac{\sin^3\theta - 1}{1-\sin^2\theta} d\theta \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta + \sin\theta + 1}{1+\sin\theta} d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \left[\sin\theta + \frac{1}{1+\sin\theta}\right] d\theta \\
&= \frac{8}{3} + \frac{8}{3}\int_0^1 \frac{1}{1+\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = \frac{16}{3}.
\end{aligned}$$

Problem 3. (15%) Use spherical coordinates to evaluate

$$\iiint_D z\,dV,$$

where the region D in the three dimensional space with coordinate (x, y, z) is described by the inequalities

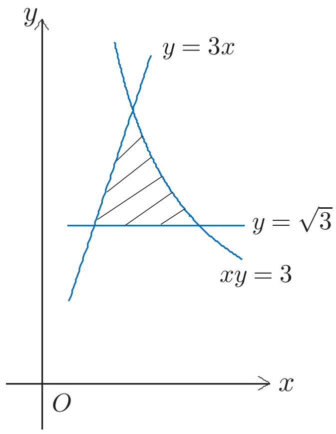
$$x^2 + y^2 + z^2 \le \sqrt{x^2+y^2}, \qquad z \ge 0$$

Sol: In spherical coordinate, the region D is bounded by $\rho^2 \le \sqrt{\rho^2\sin^2\phi} = \rho\sin\phi$ (or $\rho \le \sin\phi$) and $\rho\cos\phi \ge 0$ (or $0 \le \phi \le \pi/2$). Therefore, using the spherical coordinate,

$$\begin{aligned}
\iiint_D z\,dV &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\sin\phi} \rho\cos\phi\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{\pi}{2}\int_0^{\pi/2} \sin\phi\cos\phi\rho^4\Big|_{\rho=0}^{\rho=\sin\phi} d\phi \\
&= \frac{\pi}{2}\int_0^{\pi/2} \sin^5\phi\cos\phi\,d\phi \\
&= \frac{\pi}{2}\frac{1}{6}\sin^6\phi\Big|_{\phi=0}^{\phi=\pi/2} = \frac{\pi}{12}.
\end{aligned}$$

Problem 4. Let R be the region in the first quadrant bounded by the lines $y = 3x$, $y = \sqrt{3}$ and the hyperbola $xy = 3$ (see the figure for reference).

Compute the double integral $\iint_R xy\,dA$ in the following way:



1. (5%) Let $x = u/v$ and $y = v$. Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$.

2. (5%) Sketch the region $\widetilde{R}$ in uv plane so that every point in R corresponds to a unique point in $\widetilde{R}$. In other words, find the corresponding integral domain in the uv plane.

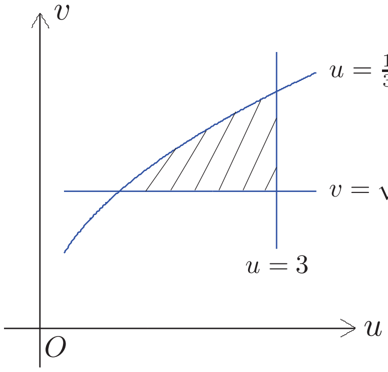
3. (10%) Convert the double integral to an integral in the uv coordinate and then compute the double integral.

Sol:

1. By definition of the Jacobian,

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = 1/v.$$

2. The line $y = \sqrt{3}$ corresponds to the line $v = \sqrt{3}$ in uv plane, while the hyperbola $xy = 3$ corresponds to the line $u = 3$. Moreover, the line $y = 3x$ corresponds to the curve $v = 3u/v$ or $u = \dfrac{1}{3}v^2$. Therefore, the region $\widetilde{R}$ on the uv plane is the region enclosed by $v = \sqrt{3}$, $u = 3$ and the parabola $u = \dfrac{1}{3}v^2$ that is plotted as follows:



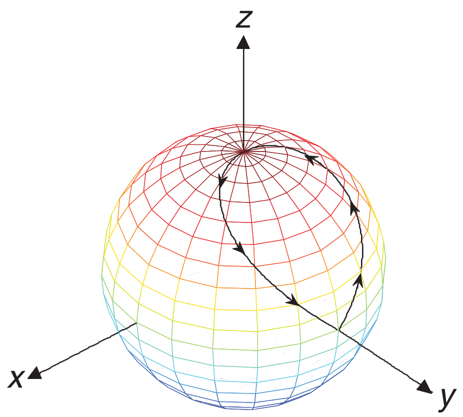
3. By the change of variable formula,

$$\iint_R xy\,dA = \iint_{\widetilde{R}} u\left|\frac{\partial(x,y)}{\partial(u,v)}\right| d\widetilde{A} = \int_{\sqrt{3}}^{3}\int_{\frac{1}{3}v^2}^{3} \frac{u}{v}\,dudv$$
$$= \int_{\sqrt{3}}^{3} \frac{u^2}{2v}\Big|_{u=\frac{1}{3}v^2}^{3} dv = \int_{\sqrt{3}}^{3}\left[\frac{9}{2v} - \frac{v^3}{18}\right]dv = \left[\frac{9}{2}\ln v - \frac{v^4}{72}\right]\Big|_{v=\sqrt{3}}^{v=3}$$
$$= \frac{9}{4}\ln 3 - 1.$$

Problem 5. Let C be a smooth curve parametrized by

$$\vec{r}\,(t) = (\cos t\sin t, \sin t\sin t, \cos t)\,, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}\,.$$

(See the figure for reference).



The curve C divides the unit sphere into two parts, and let Σ be the smaller part. Find the surface area of Σ by completing the following:

1. (5%) Projecting Σ onto the xy plane, and called the projection R. Then Σ is the corresponding surface over R; that is,

$$\Sigma = \left\{(x,y,z)\in\mathbb{R}^3 \,\Big|\, z = \sqrt{1-x^2-y^2}, (x,y)\in R\right\}.$$

Show that if (x,y) is on the boundary of R, then (x,y) satisfies

$$x^2 + y^2 - y = 0.$$

2. (15%) Find the surface area of Σ by computing $\displaystyle\iint_R dS$.

Sol:

1. Assume that $(x,y) = (\cos t\sin t, \sin t\sin t)$ is on the boundary of R. Then

$$x^2 + y^2 = \sin^2 t = \frac{y^2}{x^2+y^2} \quad \text{or} \quad (x^2+y^2)^2 = y^2.$$

Since $y \ge 0$, we find that $x^2 + y^2 = y$.

2. The surface area of Σ can be computed by

$$\int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \frac{1}{\sqrt{1-x^2-y^2}} dxdy = 2\int_0^1 \sin^{-1} \frac{\sqrt{y-y^2}}{\sqrt{1-y^2}} dy = 2\int_0^1 \tan^{-1}\sqrt{y} dy$$

$$\overset{(y=\tan^2\theta)}{=} 2\int_0^{\frac{\pi}{4}} \theta d(\tan^2\theta) = 2\Big[\theta\tan^2\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2\theta d\theta\Big]$$

$$= 2\Big[\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \sec^2\theta d\theta + \int_0^{\frac{\pi}{4}} d\theta\Big] = 2\Big[\frac{\pi}{4} - \tan\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{4}} + \frac{\pi}{4}\Big] = \pi - 2.$$