Calculus II Midterm 3 Sample

National Central University, Spring 2012, May 27, 2012

Problem 1. Evaluate the double integral $\int_0^1 \int_{e^x}^e e^x dydx$ in the following way:

1. Directly integrate by computing the iterated integral.
2. Sketch the region of integration.
3. Interchange the order of integration, and evaluate the double integral again.

Problem 2. A torus $\mathbb{T}^2$ is obtained by rotating the circle $(x-2)^2 + z^2 = 1$ (in the xz plane) about the z-axis.

1. Express the torus in the cylindrical coordinate. In other words, write down the relations between r, θ and z if $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ is on the torus.
2. Let D be the solid region bounded by $\mathbb{T}^2$. Find the corresponding domain of D in the (r, θ, z) space (Suppose D is the same as $a \le r \le b$, $\alpha \le \theta \le \beta$, and $F_1(r, \theta) \le z \le F_2(r, \theta)$, find a, b, α, β as well as F_1, F_2).
3. Find the volume of D, or $\iiint_D dV$, using the cylindrical coordinate.

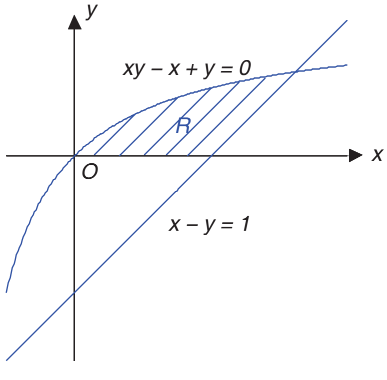
Problem 3. Use spherical coordinates to evaluate

$$\iiint_D \sqrt{x^2 + y^2 + z^2} dV,$$

where the region D in the three dimensional space with coordinate (x, y, z) is described by the inequalities

$$x^2 + y^2 + z^2 \le z\,, \quad z \ge 0\,, \quad y \ge 0\,.$$

Problem 4. Let R be the region in the first quadrant bounded by curves $xy - x + y = 0$ and $x - y = 1$ (see the figure for reference).



Compute the double integral $\iint_R x^2y^2(x+y)e^{-(x-y)^2} dA$ in the following way:

1. Let $u = xy$ and $v = x - y$. Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

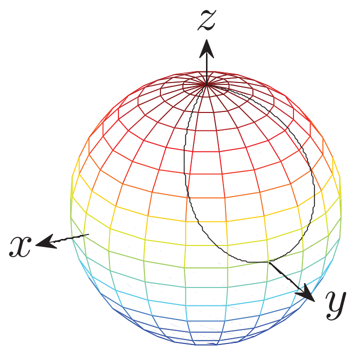
2. Sketch the region $\widetilde{R}$ in uv plane so that every point in R corresponds to a unique point in $\widetilde{R}$. In other words, find the corresponding integral domain in the uv plane.

3. Convert the double integral to an integral in the uv coordinate and then compute the double integral.

Problem 5. Let C be a smooth curve parametrized by

$$\vec{r}\,(t) = (\cos t \sin t, \cos t, \sin t \sin t)\,, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\,.$$

(See the figure for reference).



The curve C divides the unit sphere into two parts, and let Σ be the smaller part. Find the surface area of Σ by completing the following:

1. Projecting Σ onto the xy plane, and called the projection R. Then Σ is the corresponding surface over R; that is,

$$\Sigma = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, z = \sqrt{1 - x^2 - y^2}, (x,y) \in R\right\}.$$

   Show that if (x,y) is on the boundary of R, then (x,y) satisfies

$$x^2 - y^2 + y^4 = 0.$$

2. Find the surface area of Σ by computing $\displaystyle\iint_R dS$.