Calculus II Midterm 2, Sample

National Central University, Spring 2012, May. 5, 2012

Problem 1. (**Gradients and directional derivatives**) Let a function f of two variables be defined by

$$f(x,y) = \begin{cases} x+y & \text{if } x=0 \text{ or } y=0\,, \\ 0 & \text{otherwise,} \end{cases}$$

and $\vec{u} = (\cos\theta, \sin\theta)$ be a unit vector.

1. Compute $D_{\vec{u}} f(0,0)$, the directional derivative of f in the direction $\vec{u}$ at the point $(0,0)$.
2. Compute the gradient of f at the point $(0,0)$.
3. Is $D_{\vec{u}} f(0,0)$ the same as $\nabla f(0,0) \cdot \vec{u}$?
4. Does the existence of the directional derivatives of a function in every direction imply the continuity of this function? Why?

Problem 2. (**Chain rule**) Let $w = f(x,y,z)$, and (r,θ,z) be the cylindrical coordinate. Show that

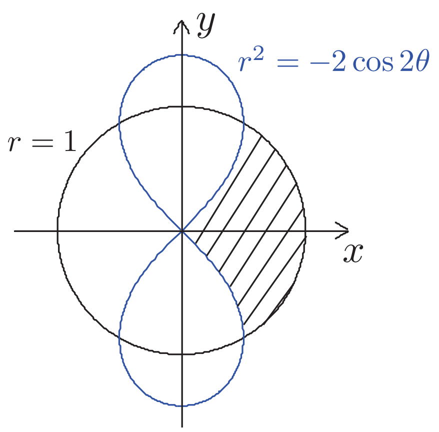
$$\|\nabla f\|^2 = \Big(\frac{\partial w}{\partial \rho}\Big)^2 + \frac{1}{r}\Big(\frac{\partial w}{\partial \theta}\Big)^2 + \Big(\frac{\partial w}{\partial z}\Big)^2 .$$

Problem 3. (**Tangent planes**) Let $F(x,y,z) = \cos(x+y^2+z^3) - e^{x^3+y^2+z}$.

1. Find the tangent plane P_1 for the zero level surface of F at the origin.
2. Let P_2 be the tangent plane for the zero level surfae of F at the point $(1,0,-1)$. Find the angle between P_1 and P_2.

Problem 4. (**Extreme values and saddle points**) Find all relative extrema and saddle points of $F(x,y) = 2xy - \frac{1}{2}(x^4+y^4)+1$.

Problem 5. (**Lagrange multiplier**) Let R be the region bounded by the circle $r=1$ and outside the lemniscate $r^2 = -2\cos 2\theta$, and is located on the right half plane (see the shaded region in the graph).



Find the minimum value of the function $f(x, y) = y^4 + 2x^2y^2 - 2y^2$ with (x, y) located in the region R.

Problem 6. (**Applications**) Let $P_1 = (0, 1)$, $P_2 = (0, 0)$, $P_3 = (0, -1)$, $P_4 = (-1, 1)$, $P_5 = (-1, 0)$, and $P_6 = (-1, -1)$ be six points on the plane.

1. Find a straight line L so that the sum of the squared distance

$$S = \sum_{i=1}^{6} \text{dist}(P_i, L)^2$$

   is smallest, where $\text{dist}(P, L)$ denotes the distance from a point P to line L.

2. Fine the least square regression line.