Calculus II Midterm 1 - Sample

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Problem 1. Evaluate the definite integral $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3+2\cos 4x}dx$. $\Big(\text{Ans} = \dfrac{\pi}{2\sqrt{5}}\Big)$

Sol. Let $u = 4x$. Then $du = 4dx$; thus

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3+2\cos 4x}dx = \frac{1}{4}\int_{-\pi}^{\pi} \frac{1}{3+2\cos u}du = \frac{1}{2}\int_{0}^{\pi} \frac{1}{3+2\cos u}du.$$

By the change of variable $t = \tan\dfrac{u}{2}$, we find that

$$\begin{aligned}\int_0^{\pi} \frac{1}{3+2\cos u}du &= \int_0^{\infty} \frac{1}{3+2\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = \int_0^{\infty} \frac{2}{t^2+5}dt \\ &= \frac{2}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}}\Big|_{t=0}^{t=\infty} = \frac{\pi}{\sqrt{5}}.\end{aligned}$$

Therefore, $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3+2\cos 4x}dx = \frac{\pi}{2\sqrt{5}}$. □

Problem 2. Find all $\alpha \in \mathbb{R}$ so that the improper integral $\displaystyle\int_1^{\infty} \frac{1}{x\big[\ln(1+x)\big]^{\alpha}}dx$ is convergent. $\Big(\text{Ans} : \alpha > 1\Big)$

Sol. Let $e^y = 1+x$. Then

$$\int_1^{\infty} \frac{1}{x\big[\ln(1+x)\big]^{\alpha}}dx = \int_{\ln 2}^{\infty} \frac{e^y dy}{(e^y-1)y^{\alpha}} = \int_{\ln 2}^{\infty} \frac{dy}{y^{\alpha}} + \int_{\ln 2}^{\infty} \frac{dy}{(e^y-1)y^{\alpha}}.$$

Therefore, by $0 \le \dfrac{1}{e^y-1} \le 1$ for $y \in [\ln 2, \infty)$,

$$\int_{\ln 2}^{\infty} \frac{dy}{y^{\alpha}} \le \int_1^{\infty} \frac{1}{x\big[\ln(1+x)\big]^{\alpha}}dx \le \int_{\ln 2}^{\infty} \frac{2dy}{y^{\alpha}};$$

thus the improper integral is convergent if and only if $\alpha > 1$.

Problem 3. Show that $\displaystyle\int_1^e (\ln x)^n dx = e\sum_{k=0}^{n-2}(-1)^k\frac{n!}{(n-k)!} + (-1)^{n-1}n!$.
Hint: Use integration by parts to show that

$$\int_1^e (\ln x)^n dx = e - n\int_1^e (\ln x)^{n-1}dx.$$

Proof. Let $u = (\ln x)^n$ and $dv = dx$ (or $v = x$). By integration by parts,

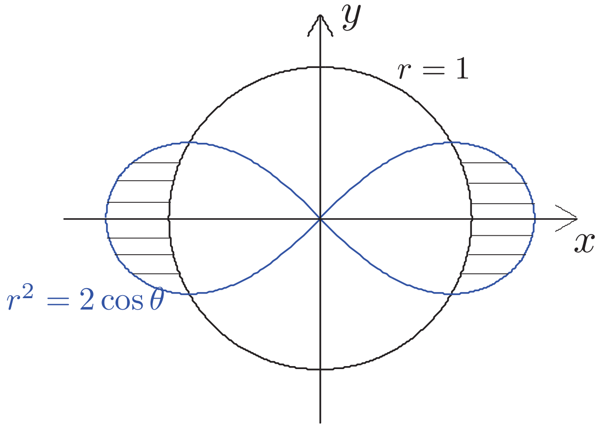
$$\int_1^e (\ln x)^n dx = x(\ln x)^n\Big|_{x=1}^{x=e} - \int_1^e xn(\ln x)^{n-1}\frac{1}{x}\,dx = e - n\int_1^e (\ln x)^{n-1}dx.$$

Therefore,

$$\begin{aligned}\int_1^e (\ln x)^n dx &= e - n\Big[e - (n-1)\int_1^e (\ln x)^{n-2}dx\Big] = e - ne + n(n-1)\int_1^e (\ln x)^{n-2}dx \\ &= e - ne + n(n-1)\Big[e - (n-2)\int_1^e (\ln x)^{n-3}dx\Big] \\ &= \cdots\cdots\cdots \\ &= e\Big[1 - n + n(n-1) - n(n-1)(n-2) + \cdots + (-1)^\ell n(n-1)\cdots(n-\ell+1)\Big] \\ &\quad + (-1)^{\ell+1} n(n-1)\cdots(n-\ell)\int_1^e (\ln x)^{n-\ell-2}dx \\ &= e\sum_{k=0}^{\ell}(-1)^k \frac{n!}{(n-k)!} + (-1)^{\ell+1}\frac{n!}{(n-\ell-1)!}\int_1^e (\ln x)^{n-\ell-1}dx.\end{aligned}$$

The conclusion can be made by letting $\ell = n-2$. □

Problem 4. Let R be the region bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$ (see the shaded region in the graph).



1. Find the area of R. $\Big(\text{Ans} = 2\sqrt{3} - \dfrac{2\pi}{3}\Big)$
2. Find the slope of the tangent line passing thought the point on the lemniscate corresponding to $\theta = \dfrac{\pi}{6}$. $\big(\text{Ans} = 0\big)$
3. Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

   (a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

   $$y^4 + a(x)y^2 + b(x) = 0 \tag{1}$$

   for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$. $\Big(\text{Ans} : a(x) = 2(x^2+1), b(x) = x^4 - 2x^2\Big)$

(b) Solving (1), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1+4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = 2 \times \left[\pi \int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} c(x)dx - \pi \int_{\frac{\sqrt{3}}{2}}^{1} d(x)dx\right].$$

Compute $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{1} \big[d(x) - (1-x^2)\big]dx$. $\big(\text{Ans} = 0\big)$

(c) Evaluate I by first computing the integral $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}dx$, and then find I.

$$\left(\text{Ans}: \begin{array}{l} \displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}dx = \pi\Big(\frac{3\sqrt{2}}{2} - \frac{\sqrt{3}}{2} + \frac{1}{4}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}}\Big), \\ \displaystyle I = \pi\Big(\frac{1}{2}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}} + \sqrt{3} - \frac{\sqrt{2}}{3} - \frac{4}{3}\Big). \end{array}\right)$$

4. Find the surface area of the surface of revolution obtained by rotating the boundary of R about the x-axis. $\big(\text{Ans} = 6\pi(2-\sqrt{3})\big)$

Sol.

1. First we find the points of intersection: let $2\cos 2\theta = 1$, then $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$; thus the points of intersection are

$$\Big(\frac{\sqrt{3}}{2}, \frac{1}{2}\Big), \Big(-\frac{\sqrt{3}}{2}, \frac{1}{2}\Big), \Big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big), \Big(\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big).$$

The area of the desired region is

$$4\int_0^{\frac{\pi}{6}} (2\cos 2\theta - 1)d\theta = 4\big(\sin 2\theta - \theta\big)\Big|_{\theta=0}^{\theta=\frac{\pi}{6}} = 4\Big(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\Big) = 2\sqrt{3} - \frac{2\pi}{3}.$$

2. If (x, y) is on the lemniscate, then

$$x^2 + y^2 = 2\Big(2\frac{x^2}{x^2+y^2} - 1\Big) = \frac{2(x^2-y^2)}{x^2+y^2}$$

which implies that

$$y^4 + 2(x^2+1)y^2 + x^4 - 2x^2 = 0.$$

Therefore,

$$y^2 = -(x^2+1) + \sqrt{(x^2+1)^2 - (x^4 - 2x^2)} = -(x^2+1) + \sqrt{1+4x^2}.$$

Therefore, the volume of the solid of revolution obtained by rotating R about the x-axis is

$$2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\Big[-(x^2+1)+\sqrt{1+4x^2}\Big]dx-2\pi\int_{\frac{\sqrt{3}}{2}}^{1}(1-x^2)dx$$
$$=2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx-2\pi\Big(\frac{x^3}{3}+x\Big)\Big|_{x=\frac{\sqrt{3}}{2}}^{x=\sqrt{2}}-2\pi\Big(x-\frac{x^3}{3}\Big)\Big|_{x=\frac{\sqrt{3}}{2}}^{x=1}$$
$$=2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx+2\pi\Big(\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{8}-\frac{2\sqrt{2}}{3}-\sqrt{2}-1+\frac{1}{3}\Big)$$
$$=2\pi\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx+2\pi\Big(\sqrt{3}-\frac{5\sqrt{2}}{3}-\frac{2}{3}\Big).$$

On the other hand, the integral can be evaluated by making a change of variable $x=\dfrac{\tan\theta}{2}$:

$$\int\sqrt{1+4x^2}\,dx=\frac{1}{2}\int\sec^3\theta d\theta=\frac{1}{2}\int\sec\theta\big(\tan^2\theta+1\big)d\theta$$
$$=\frac{1}{2}\int\tan\theta d\sec\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|$$
$$=\frac{1}{2}\Big[\tan\theta\sec\theta-\int\sec^3\theta d\theta\Big]+\frac{1}{2}\ln|\sec\theta+\tan\theta|$$
$$=-\frac{1}{2}\int\sec^3\theta d\theta+\frac{1}{2}\Big[\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\Big]$$
$$=\frac{1}{4}\Big[\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\Big]+C$$
$$=\frac{1}{4}\Big[2x\sqrt{1+4x^2}+\ln|2x+\sqrt{1+4x^2}|\Big]+C.$$

Therefore,

$$\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}}\sqrt{1+4x^2}\,dx=\frac{1}{4}\Big[6\sqrt{2}+\ln(3+2\sqrt{2})-2\sqrt{3}-\ln(2+\sqrt{3})\Big]$$
$$=\pi\Big(\frac{3\sqrt{2}}{2}-\frac{\sqrt{3}}{2}+\frac{1}{4}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}}\Big)$$

and the volume of the solid of revolution obtained by rotating R about the x-axis is

$$\pi\Big(\frac{1}{2}\ln\frac{3+2\sqrt{2}}{2+\sqrt{3}}+\sqrt{3}-\frac{\sqrt{2}}{3}-\frac{4}{3}\Big).$$

3. There are two parts of the surface: one from rotating the lemniscate and the other from rotating the sphere. The area obtained by rotating the part of the lemniscate is

$$\int 2\pi y ds=\int 2\pi r\sin\theta\sqrt{r'^2+r^2}d\theta$$
$$=2\pi\int_0^{\frac{\pi}{6}}\sqrt{2\cos 2\theta}\sin\theta\sqrt{(\sqrt{2\cos 2\theta}')^2+(\sqrt{2\cos 2\theta})^2d\theta}$$
$$=4\pi\int_0^{\frac{\pi}{6}}\sqrt{\cos 2\theta}\sin\theta\frac{1}{\sqrt{\cos 2\theta}}\,d\theta$$
$$=4\pi(-\cos\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{6}}=4\pi\Big(1-\frac{\sqrt{3}}{2}\Big).$$

The part obtained by rotating the part of the sphere is

$$\int 2\pi y ds = 2\pi \int_0^{\frac{\pi}{6}} \sin\theta \sqrt{1'^2 + 1^2} d\theta = 2\pi\Big(1 - \frac{\sqrt{3}}{2}\Big).$$

The total area is then $12\pi\Big(1 - \frac{\sqrt{3}}{2}\Big)$. □

Problem 5. Parametrize the curve

$$\mathbf{r} = \mathbf{r}(t) = \tan^{-1}\frac{t}{\sqrt{1-t^2}}\mathbf{i} + \sin^{-1} t\mathbf{j} + \cos^{-1} t\mathbf{k}, \quad t \in \Big[-1, 0.5\Big],$$

in the same orientation in terms of arc-length measured from the point where $t = 0$.

$\Big(\text{Ans} : \mathbf{r}_1 = \mathbf{r}_1(s) = \frac{s}{\sqrt{3}}\mathbf{i} + \frac{s}{\sqrt{3}}\mathbf{j} + (\frac{\pi}{2} - \frac{s}{\sqrt{3}})\mathbf{k}, \quad s \in \Big[-\frac{\sqrt{3}\pi}{2}, \frac{\sqrt{3}\pi}{6}\Big]\Big)$

Sol. By

$$\frac{d}{dt}\tan^{-1}\frac{t}{\sqrt{1-t^2}} = \frac{1}{1 + \frac{t^2}{1-t^2}} \frac{\sqrt{1-t^2} + \frac{t^2}{\sqrt{1-t^2}}}{1-t^2} = \frac{1}{\sqrt{1-t^2}}$$

and

$$\frac{d}{dt}\sin^{-1} t = \frac{1}{\sqrt{1-t^2}}, \qquad \frac{d}{dt}\cos^{-1} t = \frac{-1}{\sqrt{1-t^2}},$$

we find that

$$\begin{aligned} s(t) &= \int_0^t \Big|\frac{d\mathbf{r}}{dt}(t')\Big| dt' = \int_0^t \sqrt{\frac{1}{1-t'^2} + \frac{1}{1-t'^2} + \frac{1}{1-t'^2}}\, dt' \\ &= \sqrt{3}\int_0^t \frac{1}{\sqrt{1-t'^2}} dt' = \sqrt{3}\sin^{-1} t. \end{aligned}$$

Therefore, $t = t(s) = \sin\frac{s}{\sqrt{3}}$; thus required the arc-length parametrization is

$$\mathbf{r}_1 = \mathbf{r}_1(s) = \frac{s}{\sqrt{3}}\mathbf{i} + \frac{s}{\sqrt{3}}\mathbf{j} + (\frac{\pi}{2} - \frac{s}{\sqrt{3}})\mathbf{k}, \quad s \in \Big[-\frac{\sqrt{3}\pi}{2}, \frac{\sqrt{3}\pi}{6}\Big].$$