Calculus Homework 3

National Central University, Spring semester 2012

Problem 1. (15%) Find the volume common to two circular cylinders, both with radius r, if the axes of the cylinders intersect at right angles.

Sol: See the Textbook.

Problem 2. (25%) Complete the following.

(1) Show that for each $p > 0$, $t^{p-1} \le \mathrm{C}_p e^{t/2}$ if $t > 2p$ for some constant $\mathrm{C}_p > 0$. In particular, C_p can be chosen as

$$\mathrm{C}_p = (2p)^{p-1} e^{-p} .$$

(2) Show that the improper integral $\displaystyle\int_0^\infty e^{-t} t^{p-1} dt$ is convergent for all $p > 0$.

(3) Let $\Gamma(p) = \displaystyle\int_0^\infty e^{-t} t^{p-1} dt$. By (2) $\Gamma(p)$ is defined for all $p > 0$. Show that $\Gamma(p+1) = p\Gamma(p)$ for all $p > 0$.

Proof. (1) Let $f(t) = \dfrac{t}{2} + \ln \mathrm{C}_p - (p-1)\ln t$ with C_p defined as above. Then

$$\begin{aligned} f\big(2p\big) &= p + \ln\Big[(2p)^{p-1}e^{-p}\Big] - (p-1)\ln(2p) \\ &= p + (p-1)\ln(2p) - p - (p-1)\ln(2p) = 0 , \end{aligned}$$

and

$$f'(t) = \frac{1}{2} - \frac{p-1}{t} \ge 0 \qquad \forall\, t \ge 2p .$$

As a consequence, $f(t) \ge f(2p) = 0$ for all $t \ge 2p$; hence

$$\frac{t}{2} + \ln \mathrm{C}_p \ge (p-1)\ln t \quad \forall\, t \ge 2p \quad \Rightarrow \quad \mathrm{C}_p e^{\frac{t}{2}} \ge t^{p-1} \quad \forall\, t \ge 2p .$$

(2) Breaking the improper integral into two pieces, we find that

$$\begin{aligned} \int_0^\infty e^{-t}t^{p-1}dt &= \int_0^{2p} e^{-t}t^{p-1}dt + \int_{2p}^\infty e^{-t}t^{p-1}dt \le \int_0^{2p} t^{p-1}dt + \int_{2p}^\infty e^{-t}\mathrm{C}_p e^{\frac{t}{2}}dt \\ &= 2^p p^{p-1} + 2\mathrm{C}_p e^{2p} < \infty . \end{aligned}$$

Therefore, the improper integral $\displaystyle\int_0^\infty e^{-t}t^{p-1}dt$ is convergent for all $p > 0$.

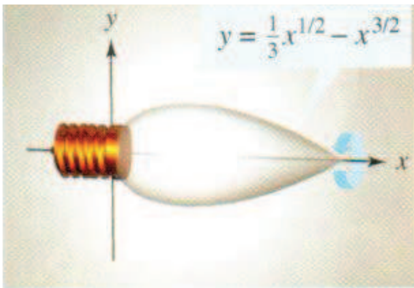
(3) Integrating by parts with $u = t^p$ and $dv = e^{-t}dt$ (or $v = -e^{-t}$), we find that

$$\Gamma(p+1) = \int_0^\infty e^{-t}t^p dt = -e^{-t}t^p\Big|_{t=0}^{t=\infty} + p\int_0^\infty e^{-t}t^{p-1}dt = p\Gamma(p) .$$

□

Problem 3. (20%)

(1) An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{0.5} - x^{1.5}$, $0 \le x \le \frac{1}{3}$, about the x-axis, where x and y are measured in feet (see figure). Find the surface area of the bulb.



(2) Find the centroid of the region bounded by the surface of the light bulb.

Sol:

(1) $y = \frac{1}{3}x^{0.5} - x^{1.5}$, $y' = \frac{1}{6}x^{-0.5} - \frac{3}{2}x^{0.5}$; hence

$$ds = \sqrt{1+y'^2}dx = \sqrt{1 + \frac{1}{36x} - \frac{1}{2} + \frac{9x}{4}} = \frac{1}{6}x^{-0.5} + \frac{3}{2}x^{0.5}.$$

Therefore, the requested surface is

$$\int_0^{\frac{1}{3}} 2\pi\big(\frac{1}{3}x^{0.5} - x^{1.5}\big)\big(\frac{1}{6}x^{-0.5} + \frac{3}{2}x^{0.5}\big)dx = 2\pi\int_0^{\frac{1}{3}}\Big[\frac{1}{18} - \frac{x}{6} + \frac{x}{2} - \frac{3}{2}x^2\Big]dx = \frac{\pi}{27}(\text{ft}^2).$$

(2) We only need to compute the x-coordinate $\bar{x}$ of the centroid since the centroid locates on the x-axis. By the formula for centroid,

$$\bar{x} = \frac{\displaystyle\int_0^{\frac{1}{3}} \pi x\big(\frac{1}{3}x^{0.5} - x^{1.5}\big)^2 dx}{\displaystyle\int_0^{\frac{1}{3}} \pi\big(\frac{1}{3}x^{0.5} - x^{1.5}\big)^2 dx} = \frac{2}{15}.$$ □

Problem 4. (15%) Evaluate the definite integral $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin x}dx$.

Sol: We first find the indefinite integral by the substitution of variable $t = \tan\frac{x}{2}$:

$$\int \frac{1}{2+\sin x}dx = \int \frac{1}{2+\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{(t+1/2)^2 + (\sqrt{3}/2)^2}$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tan\frac{x}{2}+1}{\sqrt{3}} + C.$$

Since $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin x}dx = \int_{-\pi}^{\pi} \frac{1}{2+\sin x}dx$, we find that

$$\int_0^{2\pi} \frac{1}{2+\sin x}dx = \frac{2}{\sqrt{3}} \times \Big[\tan^{-1}\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\Big]\Big|_{x=-\pi}^{x=\pi} = \frac{2}{\sqrt{3}}\Big[\frac{\pi}{2} - \big(-\frac{\pi}{2}\big)\Big] = \frac{2\pi}{\sqrt{3}}.$$ □

Problem 5. (25%) The goal of this problem is to find the indefinite integral $\displaystyle\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx$. Complete the following.

(1) By the substitution of variable $x^2 = \tan\theta$, show that

$$\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx = \frac{1}{2}\int \frac{1}{\cos\theta\sin^{\frac{1}{2}}\theta}d\theta\,.$$

(2) Then make another substitution of variable $u^2 = \sin\theta$, show that

$$\int \frac{1}{\cos\theta\sin^{\frac{1}{2}}\theta}d\theta = \int \frac{2}{(1-u^4)}du\,.$$

(3) Using the technique of integrating rational functions by partial fractions, find the indefinite integral in (1) and then express the result in terms of x so that one obtains

$$\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx = \frac{1}{2}\tan^{-1}\left[(1+x^{-4})^{-\frac{1}{4}}\right] + \frac{1}{4}\ln\frac{(x^4+1)^{\frac{1}{4}}+x}{(x^4+1)^{\frac{1}{4}}-x} + C\,.$$

Proof.

(1) Since $x^2 = \tan\theta$, $2xdx = \sec^2\theta d\theta$ or $dx = \dfrac{\sec^2\theta}{2\tan^{1/2}\theta}d\theta$; hence

$$\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx = \int \frac{1}{(1+\tan^2\theta)^{\frac{1}{4}}}\cdot\frac{\sec^2\theta}{2\tan^{1/2}\theta}d\theta = \frac{1}{2}\int \frac{\sec^{\frac{3}{2}}\theta}{\tan^{\frac{1}{2}}\theta}d\theta = \frac{1}{2}\int \frac{1}{\cos\theta\tan^{\frac{1}{2}}\theta}d\theta\,.$$

(2) Since $u^2 = \sin\theta$, $2udu = \cos\theta d\theta$; hence

$$\int \frac{1}{\cos\theta\sin^{\frac{1}{2}}\theta}d\theta = \int \frac{\cos\theta}{(1-\sin^2\theta)\sin^{\frac{1}{2}}\theta}d\theta = \int \frac{2u}{(1-u^4)u}du = \int \frac{2}{1-u^4}du\,.$$

(3) By partial fractions,

$$\begin{aligned}\int \frac{2}{1-u^4}du &= \int \left[\frac{1}{1+u^2} + \frac{1}{1-u^2}\right]du = \tan^{-1}u + \frac{1}{2}\int \left[\frac{1}{1-u} + \frac{1}{1+u}\right]du \\ &= \tan^{-1}u - \frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u) + C \\ &= \tan^{-1}u + \frac{1}{2}\ln\frac{1+u}{1-u} + C\,.\end{aligned}$$

In order to express the indefinite integral in terms of x , we need to find the relation between u and x . Since $x^2 = \tan\theta$ and $u^2 = \sin\theta$, $u = (1+x^{-4})^{\frac{-1}{4}}$, thus

$$\begin{aligned}\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx &= \frac{1}{2}\left[\tan^{-1}(1+x^{-4})^{-\frac{1}{4}} + \frac{1}{2}\ln\frac{1+(1+x^{-4})^{-\frac{1}{4}}}{1-(1+x^{-4})^{-\frac{1}{4}}}\right] + C \\ &= \frac{1}{2}\tan^{-1}\left[(1+x^{-4})^{-\frac{1}{4}}\right] + \frac{1}{4}\ln\frac{(x^4+1)^{\frac{1}{4}}+x}{(x^4+1)^{\frac{1}{4}}-x} + C\,.\end{aligned}$$

□