Calculus Homework 2

National Central University, Spring semester 2012

Problem 1. (10%) Find $\dfrac{d}{dx}\displaystyle\int_{\ln x}^{\tan^{-1}x} 3^{-u^2}du$ for $x > 0$.

Sol: By the fundamental theorem of Calculus,

$$\frac{d}{dx}\int_{\ln x}^{\tan^{-1}x} 3^{-u^2}du = 3^{-(\tan^{-1}x)^2}\frac{d}{dx}\tan^{-1}x - 3^{-(\ln x)^2}\frac{d}{dx}\ln x = \frac{3^{-(\tan^{-1}x)^2}}{1+x^2} - \frac{3^{-(\ln x)^2}}{x}. \qquad \square$$

Problem 2. (10%) Find $\dfrac{d}{dx}\ln\big[\sin^{-1}(e^{x^2})+2\big]$ for $x \in \mathbb{R}$.

Sol: By the chain rule,

$$\begin{aligned}\frac{d}{dx}\ln\big[\sin^{-1}(e^{x^2})+2\big] &= \frac{1}{\sin^{-1}(e^{x^2})+2}\cdot\frac{d}{dx}\big[\sin^{-1}(e^{x^2})+2\big] = \frac{1}{\sin^{-1}(e^{x^2})+2}\cdot\frac{1}{\sqrt{1-e^{2x^2}}}\frac{d}{dx}e^{x^2} \\ &= \frac{2xe^{x^2}}{\big[\sin^{-1}(e^{x^2})+2\big]\cdot\sqrt{1-e^{2x^2}}}. \qquad \square\end{aligned}$$

Problem 3. (10%) Find $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\frac{\pi}{2}-x)^{\cot x}$.

Sol: By $f(x)^{g(x)} = e^{g(x)\ln f(x)}$,

$$\lim_{x\to\frac{\pi}{2}^-}(\frac{\pi}{2}-x)^{\cot x} = \lim_{x\to\frac{\pi}{2}^-} e^{\cot x\ln(\frac{\pi}{2}-x)} = \exp\Big(\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\frac{\pi}{2}-x)}{\tan x}\Big).$$

The limit of the exponent is indeterminate form of type $\frac{\infty}{\infty}$, and we apply the L'Hospital rule to obtain that

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\frac{\pi}{2}-x)}{\tan x} &= \lim_{x\to\frac{\pi}{2}^-}\frac{1}{(x-\frac{\pi}{2})\sec^2 x} = \lim_{x\to\frac{\pi}{2}^-}\frac{\cos^2 x}{(x-\frac{\pi}{2})} \\ &= \lim_{x\to\frac{\pi}{2}^-}\frac{-2\cos x\sin x}{1} = 0;\end{aligned}$$

hence $\displaystyle\lim_{x\to\frac{\pi}{2}^-}(\frac{\pi}{2}-x)^{\cot x} = e^0 = 1$. $\square$

Problem 4. (15%) Find the indefinite integral $\displaystyle\int x^2(\ln x)^2dx$. Verify your answer by differentiating the result you obtain.

Sol: Let $u = (\ln x)^2$ and $dv = x^2dx$ (or $v = \frac{1}{3}x^3$), then

$$\begin{aligned}\int x^2(\ln x)^2dx &= \frac{1}{3}x^3(\ln x)^2 - \int\frac{1}{3}x^3\cdot 2\ln x\cdot\frac{1}{x}dx \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\int x^2\ln x dx \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{3}\Big[\frac{1}{3}x^3\ln x - \int\frac{1}{3}x^3\cdot\frac{1}{x}dx\Big] \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{9}\int x^2dx \\ &= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C.\end{aligned}$$

Differentiating the right-hand side,

$$\begin{aligned}\frac{d}{dx}\Big[\frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C\Big] \\ &= x^2(\ln x)^2 + \frac{1}{3}x^3\cdot 2\ln x\cdot\frac{1}{x} - \frac{2}{3}x^2\ln x - \frac{2}{9}x^3\cdot\frac{1}{x} + \frac{2}{9}x^2 \\ &= x^2(\ln x)^2 + \frac{2}{3}x^2\ln x - \frac{2}{3}x^2\ln x - \frac{2}{9}x^2 + \frac{2}{9}x^2 \\ &= x^2(\ln x)^2\,.\end{aligned}$$

So $\frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C$ is the anti-derivative of $x^2(\ln x)^2$. □

Problem 5. (10%) Find the definite integral $\int_0^{\frac{\pi}{4}}\tan^4 x dx$.

Sol: Let $u = \tan x$. Then $du = \sec^2 x dx$; thus

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\tan^4 x dx &= \int_0^{\frac{\pi}{4}}\tan^2 x(\sec^2 x - 1)dx = \int_0^1 u^2 du - \int_0^{\frac{\pi}{4}}\tan^2 x dx \\ &= \frac{1}{3} - \int_0^{\frac{\pi}{4}}(\sec^2 x - 1)dx = \frac{1}{3} + \frac{\pi}{4} - \tan x\Big|_{x=0}^{x=\frac{\pi}{4}} = \frac{\pi}{4} - \frac{2}{3}\,.\end{aligned}$$ □

Problem 6. (25%) Find the indefinite integral $\int \sin^2 x dx$ using

(1) The half angle formula $\sin^2 x = \dfrac{1-\cos 2x}{2}$;

(2) Using the technique of integration by parts with $u = \sin x$ and $dv = \sin x dx$;

(3) Using the substitution of variable $t = \tan\frac{x}{2}$ and transform the original integral into the integral of a rational function, and use the technique of partial fractions.

Hint: For (3), you will need the recursive formula

$$\int \frac{1}{(1+x^2)^n}dx = \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int\frac{1}{(1+x^2)^{n-1}}dx \quad \forall\, n \geq 2\,.$$

Sol:

(1) By the identity $\sin^2 x = \dfrac{1-\cos 2x}{2}$,

$$\int \sin^2 dx = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{\sin 2x}{4} + C\,.$$

(2) Let $u = \sin x$ and $v = -\cos x$. Then

$$\begin{aligned}\int \sin^2 dx &= -\sin x\cos x + \int\cos^2 x dx = -\sin x\cos x + \int(1-\sin^2 x)dx \\ &= x - \sin x\cos x - \int\sin^2 x dx\,.\end{aligned}$$

Therefore,

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin x\cos x}{2} + C\,.$$

(3) Let $t=\tan\frac{x}{2}$. Then $\sin x=\frac{2t}{1+t^2}$ and $dx=\frac{2dt}{1+t^2}$. Therefore,

$$\begin{aligned}\int \sin^2 x dx &= \int \Big(\frac{2t}{1+t^2}\Big)^2 \frac{2dt}{1+t^2} = \int \frac{8t^2}{(1+t^2)^3}dt = 8\int \frac{1}{(1+t^2)^2}dt - 8\int \frac{1}{(1+t^2)^3}dt \\ &= 8\int \frac{1}{(1+t^2)^2}dt - 8\Big[\frac{t}{4(1+t^2)^2} + \frac{3}{4}\int \frac{1}{(1+t^2)^2}dt\Big] \\ &= -\frac{2t}{(1+t^2)^2} + 2\Big[\frac{t}{2(1+t^2)} + \frac{1}{2}\int \frac{1}{1+t^2}dt\Big] \\ &= \tan^{-1} t + \frac{t}{1+t^2} - \frac{2t}{(1+t^2)^2} + C \\ &= \frac{x}{2} + \frac{\sin x}{2} - \sin x \cos^2 \frac{x}{2} + C \\ &= \frac{x}{2} + \frac{\sin x}{2} - \sin x \frac{1+\cos x}{2} + C \\ &= \frac{x}{2} - \frac{\sin x \cos x}{2} + C\,.\end{aligned}$$

Problem 7. (20%) The goal of this problem is to find the indefinite integral $\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx$. Complete the following.

(1) By the substitution of variable $1+x^{-4}=u^4$, show that

$$\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx = -\int \frac{u^2}{u^4-1}du\,.$$

(2) Using the technique of integrating rational functions by partial fractions, find the indefinite integral in (1) and then express the result in terms of x so that one obtains

$$\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx = -\frac{1}{2}\tan^{-1}\big[(1+x^{-4})^{\frac{1}{4}}\big] + \frac{1}{4}\ln\frac{(x^4+1)^{\frac{1}{4}}+x}{(x^4+1)^{\frac{1}{4}}-x} + C\,.$$

Proof.

(1) Since $1+x^{-4}=u^4$, $-4x^{-5}dx=4u^3du$; thus $dx=-x^5u^3du$. Moreover, $1+x^4=x^4(1+x^{-4})=x^4u^4$; thus $(1+x^4)^{1/4}=xu$. Therefore,

$$\frac{1}{(1+x^4)^{\frac{1}{4}}}dx = -\frac{x^5u^3}{xu}du = -x^4u^2du = -\frac{u^2}{u^4-1}du\,;$$

hence

$$\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx = -\int \frac{u^2}{u^4-1}du\,.$$

(2) By $u^4-1=(u^2+1)(u^2-1)$,

$$\begin{aligned}-\frac{u^2}{u^4-1} &= -\frac{1}{2}\Big[\frac{1}{u^2+1}+\frac{1}{u^2-1}\Big] = -\frac{1}{2}\Big[\frac{1}{u^2+1} - \frac{1}{2}\Big(\frac{1}{u-1}-\frac{1}{u+1}\Big)\Big] \\ &= -\frac{1}{2}\frac{1}{(u^2+1)} - \frac{1}{4}\frac{1}{u-1} + \frac{1}{4}\frac{1}{u+1}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}-\int \frac{u^2}{u^4-1}du &= -\frac{1}{2}\tan^{-1}u - \frac{1}{4}\ln|u-1| + \frac{1}{4}\ln|u+1| + C \\ &= -\frac{1}{2}\tan^{-1}u + \frac{1}{4}\ln\frac{|u+1|}{|u-1|} + C\,;\end{aligned}$$

thus

$$\begin{aligned}\int \frac{1}{(1+x^4)^{\frac{1}{4}}}dx &= -\frac{1}{2}\tan^{-1}\left[(1+x^{-4})^{\frac{1}{4}}\right] + \frac{1}{4}\ln\frac{(1+x^{-4})^{\frac{1}{4}}+1}{(1+x^{-4})^{\frac{1}{4}}-1} + C \\ &= -\frac{1}{2}\tan^{-1}\left[(1+x^{-4})^{\frac{1}{4}}\right] + \frac{1}{4}\ln\frac{(x^4+1)^{\frac{1}{4}}+x}{(x^4+1)^{\frac{1}{4}}-x} + C\,.\end{aligned}$$

□