Calculus Homework 1

National Central University, Spring semester 2012

Problem 1. (10%) Compute the following limits.

(1) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{|x|}$ (2) $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{1-\cos x}$.

Sol.

(1) In class we have shown that $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$. Therefore,

$$\lim_{x\to 0}\frac{1-\cos x}{|x|}=\lim_{x\to 0}\left[\frac{1-\cos x}{x^2}|x|\right]=\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0}|x|=\frac{1}{2}\cdot 0=0\,.$$

(2) Since $a^3-b^3=(a-b)(a^2+ab+b^2)$,

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{1-\cos x}&=\lim_{x\to 0}\frac{(1+x^2)-1}{(1-\cos x)(\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1)}\\&=\lim_{x\to 0}\frac{x^2}{1-\cos x}\cdot\lim_{x\to 0}\frac{1}{(\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1)}\\&=2\cdot\frac{1}{3}=\frac{2}{3}\,.\end{aligned}$$

□

Problem 2. Complete the following.

(1) (5%) Let f and g be two functions, and $f(a)=g(a)=0$ for some number $a\in\mathbb{R}$. Suppose that f and g are differentiable at a, and $g'(a)\neq 0$. Show that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}\,.$$

The same conclusion can be drawn if the limit is changed to the right-hand limit or the left-hand limit, as long as f and g are differentiable from the right or the left at 0.

(2) (10%) Use (1) to compute the following limits:

(a) $\displaystyle\lim_{x\to 1}\frac{x^{\frac{3}{2}}-1}{\sin(\pi x)}$ (b) $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-1}{1-\cos x}$.

(3) (5%) Suppose that f is twice continuously differentiable. Use (1) to show that

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}=f''(a)\,.$$

Hint: For (2b), you cannot simply assign $g(x)=1-\cos x$ since $g'(0)=0$ which is not allowed in order to apply (1). However, you can make a slight modification of the limit by letting $y=x^2$. Then the original limit becomes the limit of some function of y as $y\to 0^+$. You will have to use similar technique to compute the limit in (3).

Proof.

(1) Since $f(a) = g(a) = 0$, f and g are differentiable at a, and $g'(a) \neq 0$, by the properties of limits, we find that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

(2) (a) Let $f(x) = \sqrt[3]{1+x^2} - 1$ and $g(x) = \sin x$. Then $f'(0) = \frac{2x}{3\sqrt[3]{(1+x^2)^2}}\Big|_{x=0} = 0$ and $g'(0) = \cos 0 = 1$; thus

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x^2}-1}{\sin x} = 0.$$

(b) Let $y = x^2$. Then $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x^2}-1}{1-\cos x} = \lim\limits_{y\to 0^+} \dfrac{\sqrt[3]{1+y}-1}{1-\cos\sqrt{y}}$. Let $f(y) = \sqrt[3]{1+y} - 1$ and $g(y) = 1 - \cos\sqrt{y}$. Then $f'(0) = \dfrac{1}{3}(1+y)^{-\frac{2}{3}}\Big|_{y=0} = 1/3$ and

$$g'(0) = \lim_{h\to 0^+} \frac{1-\cos\sqrt{h}}{h} = \lim_{h\to 0} \frac{1-\cos k}{k^2} = \frac{1}{2}.$$

Therefore,

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x^2}-1}{1-\cos x} = \frac{2}{3}.$$

(3) Let $k = h^2$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-2f(a)+f(a-h)}{h^2} &= \lim_{k\to 0^+} \frac{f(a+\sqrt{k})-2f(a)+f(a-\sqrt{k})}{k} \\
&= \lim_{k\to 0^+} \frac{\frac{f'(a+\sqrt{k})}{2\sqrt{k}} - \frac{f'(a-\sqrt{k})}{2\sqrt{k}}}{1} = \lim_{k\to 0^+} \frac{f'(a+\sqrt{k})-f'(a-\sqrt{k})}{2\sqrt{k}} \\
&= \lim_{h\to 0} \frac{f'(a+h)-f'(a-h)}{2h} = \frac{1}{2}\lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h} + \frac{1}{2}\lim_{h\to 0} \frac{f'(a)-f'(a-h)}{h} \\
&= \frac{1}{2}f''(a) + \frac{1}{2}f''(a) = f''(a).
\end{aligned}$$

□

Problem 3. (10%) Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} (1-\cos x)\sin(\cot x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ Find the derivatives of f.

Sol. If $x \neq 0$,

$$\begin{aligned}
f'(x) &= (1-\cos x)'\sin(\cot x) + (1-\cos x)\big[\sin(\cot x)\big]' \\
&= \sin x \sin(\cot x) + (1-\cos x)\cos(\cot x)\big[\cot x\big]' \\
&= \sin x \sin(\cot x) - (1-\cos x)\cot x \csc x \cos(\cot x).
\end{aligned}$$

The derivative of f at 0 is $f'(0) = \lim\limits_{h\to 0} \dfrac{(1-\cos h)\sin(\cot h)}{h}$. However, since

$$-\frac{1-\cos h}{|h|} \le \frac{(1-\cos h)\sin(\cot h)}{h} \le \frac{1-\cos h}{|h|} \quad \text{if } h \neq 0\,,$$

by the Squeeze Theorem and Problem 1(1), we find that $f'(0) = 0$. Therefore,

$$f'(x) = \begin{cases} \sin x \sin(\cot x) - (1-\cos x)\cot x \csc x \cos(\cot x) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

□

Problem 4. Suppose that $f : (0,\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to (0,\infty)$ are two strictly increasing, differentiable functions satisfying

$$f(g(x)) = x \quad \forall\, x \in \mathbb{R}\,, \qquad g(f(x)) = x \quad \forall\, x \in (0,\infty)\,,$$

and $f(ab) = f(a) + f(b)$ for all $a, b > 0$.

(1) (5%) Show that $f'(x) = \dfrac{f'(1)}{x}$ for all $x > 0$.

(2) (5%) Show that $f'(1)g'(0) = 1$.

(3) (5%) Show that $g'(x) = g(x)g'(0)$ for all $x \in \mathbb{R}$.

Proof.

(1) For any $c > 0$, $f(cx) = f(c) + f(x)$. Differentiate both sides with respect to x, we find that

$$cf'(cx) = f'(x) \qquad \forall\, x > 0\,.$$

This relation holds for all fixed $c > 0$; hence in particular letting $c = 1/x$ we conclude that $f'(x) = \dfrac{f'(1)}{x}$ for all $x > 0$.

(2) First of all, we observe that $f(1\cdot 1) = f(1) + f(1)$; hence $f(1) = 0$. By letting $x = 1$ in the relation $g(f(x)) = x$, we obtain that $g(0) = 1$. By differentiating $f(g(x)) = x$, we find that

$$f'(g(x))g'(x) = 1\,.$$

Letting $x = 0$ in the relation above gives us $f'(1)g'(0) = 1$.

(3) Similarly, since $g(f(x)) = x$,

$$g'(f(x))f'(x) = 1 \quad \Rightarrow \quad g'(f(x)) = \frac{x}{f'(1)} = g'(0)x\,.$$

Replacing x by $g(x)$ in the relation above, we then conclude that $g'(x) = g'(0)g(x)$. □

Problem 5. Suppose that x and y satisfy the relation $y\sin(x^2) = x\sin(y^2)$.

(1) (5%) Find $\dfrac{dy}{dx}$ using the implicit differentiation.

(2) (5%) Find the tangent line to the curve at the point $(1,0)$.

Sol.

(1) Differentiate both sides of $y\sin(x^2) = x\sin(y^2)$ with respect to x, we find that

$$\begin{aligned} \sin(x^2)\frac{dy}{dx} + 2xy\cos(x^2) &= \sin(y^2) + 2xy\cos(y^2)\frac{dy}{dx} \\ \Rightarrow \Big[\sin(x^2) - 2xy\cos(y^2)\Big]\frac{dy}{dx} &= \sin(y^2) - 2xy\cos(x^2) \\ \Rightarrow \frac{dy}{dx} &= \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}. \end{aligned}$$

(2) At $(1,0)$, $\dfrac{dy}{dx} = 0$; hence the tangent line to the curve at $(1,0)$ is $y = 0$. □

Problem 6. Complete the following.

(1) (5%) Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b) with $|f'(x)| \le M$ for all $x \in (a,b)$. Show that

$$|f(x) - f(y)| \le M|x-y| \qquad \forall\, x, y \in [a,b].$$

(2) (5%) Suppose that $f(x) = \big(2 - \frac{\pi}{4}\big)\sin x - x\cos x$ is defined on the interval $\Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]$. Use (1) to show that

$$|f(x) - f(y)| \le \frac{\pi}{2}|x-y| \qquad \forall\, x, y \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big].$$

(3) (7%) Sketch the graph of f defined in (2) with the information of

(a) intercepts;

(b) interval of increase and decrease;

(c) extreme values and critical points; and

(d) concavity and inflection points.

Proof. (1) By the mean value theorem, there exists z between x and y such that

$$f(x) - f(y) = f'(z)(x-y) \quad \Rightarrow \quad |f(x) - f(y)| = |f'(z)||x-y| \le M|x-y|.$$

(2) It suffices to show that $|f'(x)| \le \frac{\pi}{2}$ for all $x \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)$. First of all,

$$f'(x) = \big(2 - \frac{\pi}{4}\big)\cos x - \cos x + x\sin x = \big(1 - \frac{\pi}{4}\big)\cos x + x\sin x.$$

The extreme values of $|f|$ can be obtained by the extreme values of f. In order to find the extreme values of f', we compute the second derivative of f and obtain that

$$f''(x) = -\big(1 - \frac{\pi}{4}\big)\sin x + \sin x + x\cos x = \frac{\pi}{4}\sin x + x\cos x;$$

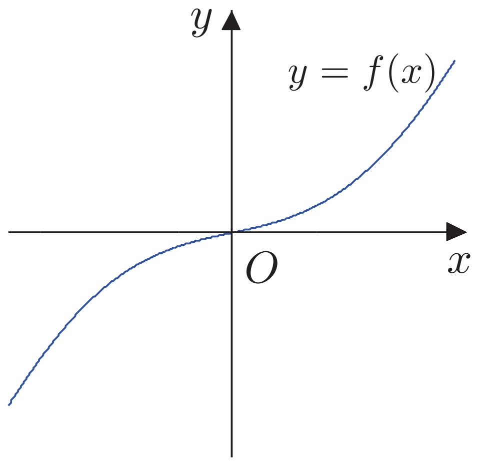
hence the critical points of f' satisfies $\frac{\pi}{4}\sin x + x\cos x = 0$. In $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, there is only one critical point which is 0. Comparing the values of f' at the critical point and the endpoints, we find that the absolute maximum of $|f'(x)|$ occurs at $-\frac{\pi}{2}$ or $\frac{\pi}{2}$; thus

$$|f'(x)| \le \left|f'\left(\frac{\pi}{2}\right)\right| = \frac{\pi}{2} \qquad \forall\, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

(3) There is only one intercept (0,0) since the only solution to $\tan x = x/c$ for $c > 1$ in $\left(-\pi/2, \pi/2\right)$ is 0. There is no critical point since there is no solution to $\tan x = c/x$ in $\left(-\pi/2, \pi/2\right)$ if $c < 0$. Moreover, $f' > 0$ in $\left(-\pi/2, \pi/2\right)$; hence f is increasing in the interval of interest. $(0,0)$ is the inflection point since f'' changes sign at $x = 0$. $f'' > 0$ if $0 < x < \pi/2$, and $f'' < 0$ if $-\pi/2 < x < 0$. Therefore, the graph is concave upward in $(0, \pi/2)$, and concave downward in $(-\pi/2, 0)$. In a nutshell, we have the following table

x	$-\pi/2$	0	$\pi/2$
f	$-(2-\pi/4)$	0	$(2-\pi/4)$
f'	$+$	$+$	$+$
f''	$-$	0	$+$

and the graph of f is



□

Problem 7. (10%) **Price elasticity of demand** ($\mathbf{E_d}$) is a measure used in economics to show the responsiveness, or elasticity, of the quantity demanded of a good or service to a change in its price. More precisely, it gives the percentage change in quantity demanded in response to a one percent change in price (holding constant all the other determinants of demand, such as income). For example, if 1% change of the price of certain good results in 1.5% change of the quantity demand of that good, then the price elasticity of demand is 1.5.

The formula for the coefficient of price elasticity of demand for a good is

$$\mathbf{E_d} = -\frac{\%\text{ change in quantity demand}}{\%\text{ change in price}} = -\frac{\Delta \mathrm{Q_d}/\mathrm{Q_d}}{\Delta \mathrm{P}/\mathrm{P}},$$

where $\mathrm{Q_d}$ is the quantity demand, and P is the price. The point price elasticity is defined using Calculus by

$$\mathbf{E_d} = -\lim_{\Delta \mathrm{Q_d} \to 0} \frac{\Delta \mathrm{Q_d}/\mathrm{Q_d}}{\Delta \mathrm{P}/\mathrm{P}} = -\frac{\mathrm{P}}{\mathrm{Q_d}} \times \frac{1}{\mathrm{P}'(\mathrm{Q_d})},$$

here we treat the price as a function of the quantity demand, and common sense suggests that an increase of quantity demand results in price drop, that is, $P'(Q_d) < 0$ for all $Q_d > 0$.

Elasticities of demand are interpreted as follows:

(1) **Perfectly inelastic** demand if $\mathbf{E_d} = 0$;

(2) Inelastic or **relatively inelastic** demand if $0 < \mathbf{E_d} < 1$;

(3) **Unit elastic**, unit elasticity, unitary elasticity, or unitarily elastic demand if $\mathbf{E_d} = 1$;

(4) Elastic or **relatively elastic** demand if $1 < \mathbf{E_d} < \infty$;

(5) **Perfectly elastic** demand if $\mathbf{E_d} = \infty$.

Show that the total revenue is maximized at the combination of price and quantity demanded where the elasticity of demand is unitary.

Hint: The total revenue function R is defined by $Q_d \times P$.

問題七翻譯：

價格需求彈性（Price elasticity of demand, E_d）在經濟學裡是用來衡量需求的數量（quantity demand）隨商品價格的變動之敏感度。更精確地說，價格需求彈性給出了 1% 的價格變動所對應的商品需求數量變動的百分比。例如假設某物價格變動的幅度 1% 引起需求量變動 1.5% 時，則價格彈性就是 1.5。價格需求彈性的公式是：

$$E_d = -\frac{\text{商品需求變動的百分比}}{\text{價格變動的百分比}} = -\frac{\Delta Q_d/Q_d}{\Delta P/P},$$

其中 Q_d 是商品需求的數量，而 P 是商品價格。點價格需求彈性（point price elasticity）是將上式取極限之後的結果：

$$E_d = -\lim_{\Delta Q_d \to 0} \frac{\Delta Q_d/Q_d}{\Delta P/P} = -\frac{P}{Q_d} \times \frac{1}{P'(Q_d)},$$

這裡我們把價格看成是商品需求量的函數，而常識告訴我們當需求上升時一定是價格下降的結果，所以對所有的 Q_d 而言，$P'(Q_d) < 0$。

對於商品的不同彈性程度，我們有不同的稱呼：

一、當 $E_d = 0$：完全無彈性；

二、當 $0 < E_d < 1$：缺乏彈性、不富彈性或無彈性；

三、當 $E_d = 1$：單位彈性、單一彈性

四、當 $1 < E_d < \infty$：富有彈性；

五、當 $E_d = \infty$：完全彈性或完全有彈性。

證明販賣該商品的總收益最大時是在價格與商品需求的組合滿足單位彈性時發生。

提示：假設 $Q_d = x$，那麼總收益函數 $R(x) = xP(x)$。

Proof. The total revenue function $R(x)$ is the same as $xP(x)$; thus

$$R'(x) = P(x) + xP'(x).$$

At the critical point x_0, $\mathrm{P}(x_0) = -x_0\mathrm{P}'(x_0)$; hence $\mathbf{E_d} = 1$ at x_0. □

Problem 8. (8%) Show that $\cos x = 2x$ has exactly one solution, and use Newton's method to compute the approximated solution x_2 with the initial guess $x_0 = 0$.

Proof. Let $f(x) = \cos x - 2x$. Then $f(-1) > 0$ while $f(1) < 0$. By the immediate value theorem we know that there exists $x \in [-1, 1]$ such that $f(x) = 0$.

Suppose there are two distinct solutions a and b with $a < b$ (that is, $f(a) = f(b) = 0$). By the Rolle theorem, there exists $a < c < b$ such that $f'(c) = 0$. However, $f'(c) = -\sin c - 2 < 0$ for all $c \in \mathbb{R}$; hence it is impossible to have two solutions.

Newton's method gives us the scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + \frac{\cos x_n - 2x_n}{\sin x_n + 2}$$

to compute approximated solutions to $f(x) = 0$. When $x_0 = 0$, then $x_1 = 1/2$; thus

$$x_2 = \frac{1}{2} + \frac{\cos 0.5 - 1}{\sin 0.5 + 2}.$$

□

Problem 9. (5%) Find an anti-derivative $G(x)$ of $g(x) = x \sin x$ satisfying $G\big(\frac{\pi}{2}\big) = 0$.
Hint: Check the derivative of f defined in Problem 1 (2).

Sol: If $f(x) = -\big(2 + \frac{\pi}{4}\big)\sin x + x\cos x$,

$$f'(x) = -\Big(2 + \frac{\pi}{4}\Big)\cos x + \cos x - x\sin x = -\Big(1 + \frac{\pi}{4}\Big)\cos x - x\sin x .$$

Therefore,

$$f'(x) + G'(x) = -\Big(1 + \frac{\pi}{4}\Big)\cos x .$$

However, an anti-derivative of the right-hand side of the equation above is $-\big(1 + \frac{\pi}{4}\big)\sin x$; hence

$$f(x) + G(x) = -\Big(1 + \frac{\pi}{4}\Big)\sin x + C$$

for some constant C. This suggests that

$$G(x) = -\Big(1 + \frac{\pi}{4}\Big)\sin x - f(x) + C = \sin x - x\cos x + C .$$

Since $G\big(\frac{\pi}{2}\big) = 0$, $C = -1$; thus $G(x) = -\big(1 + \frac{\pi}{4}\big)\sin x - f(x) + C = \sin x - x\cos x - 1$. □