Basic Mathematics（基礎數學）MA1015A Final Exam

National Central University, Jun. 21 2019

Problem 1. (20%) **True** or **False**（是非題）：每題兩分，答對得兩分，答錯倒扣兩分（倒扣至本大題零分為止）

T 1. If A is finite, then A is not equivalent to any of its proper subsets.

T 2. A non-empty subset of a countable set is countable.

T 3. The collection of functions from a finite set A to a countable set B is countable.

F 4. If $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}\backslash\{0\}$ converges to L, then $\{y_n\}_{n=1}^{\infty}$ given by $y_n = 1/x_n$ converges to $1/L$.

F 5. If $a < x_n < b$ and $\lim\limits_{n\to\infty} x_n = x$, then $a < x < b$.

T 6. A convergence sequence is bounded.

F 7. Let f, g be functions such that $g \circ f$ is defined. If $g \circ f$ is continuous, then f is continuous.

F 8. Let f, g be functions such that $g \circ f$ is defined. If $\lim\limits_{x\to a} f(x) = b$ and $\lim\limits_{y\to b} g(y) = c$, then

$$\lim_{x\to a}(g \circ f)(x) = c\,.$$

T 9. Convergent sequences of real numbers has a least upper bound.

F 10. Divergent sequences of real numbers cannot have a least upper bound.

Problem 2. Write down the definition of the following terminologies.

1. (5%) Denumerable sets.
2. (5%) Countable sets.
3. (5%) An ordered field (assuming that you know the concept of fields and partial orders).
4. (5%) Complete ordered field.

Problem 3. (15%) Show that $(0, 1)$ is uncountable.

Problem 4. (15%) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists $N > 0$ such that $|x_n - x_m| < \varepsilon$ for all $n, m > N$. Show that a convergent sequence is a Cauchy sequence.

Proof. Suppose that $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a convergent sequence with limit L. Let $\varepsilon > 0$ be given. Then there exists $N > 0$ such that

$$n \geqslant N \Rightarrow |x_n - L| < \frac{\varepsilon}{2}\,.$$

Therefore, if $n, m \geqslant N$,

$$|x_n - x_m| \leqslant |x_n - L| + |x_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

□

Problem 5. (15%) Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ be a function. Show that f is continuous at a if and only if for every sequence $\{x_n\}_{n=1}^{\infty} \subseteq I$ converging to a, one has $\lim_{n\to\infty} f(x_n) = f(a)$.

Proof. Note that $g : I \to \mathbb{R}$ is continuous at $a \in I$ if and only if

$$(\forall\, \epsilon > 0)(\exists\, \delta > 0)\big(|x-a| < \delta \wedge x \in I \Rightarrow \big|g(x) - g(a)\big| < \varepsilon\big)\,.$$

($\Rightarrow$) Let $\{x_n\}_{n=1}^{\infty} \subseteq I$ be a convergent sequence with limit a, and $\varepsilon > 0$ be given. Since f is continuous at a, there exists $\delta > 0$ such that

$$|x-a| < \delta \wedge x \in I \Rightarrow \big|f(x) - f(a)\big| < \varepsilon\,.$$

Since $\lim_{n\to\infty} x_n = a$, there exists $N > 0$ such that

$$n \geqslant N \Rightarrow |x_n - a| < \delta\,.$$

Therefore, if $n \geqslant N$, then by the fact that $x_n \in I$ for all $n \in \mathbb{N}$, we have

$$\big|f(x_n) - f(a)\big| < \varepsilon\,.$$

($\Leftarrow$) Suppose the contrary that f is not continuous at a. Then there exists $\varepsilon > 0$ such that

$$(\forall\, \delta > 0)\big(|x-a| < \delta \wedge x \in I \wedge \big|f(x) - f(a)\big| \geqslant \varepsilon\big)\,.$$

In particular, for each $\delta = \dfrac{1}{n}$ with $n \in \mathbb{N}$, there exists $x_n \in I$ satisfying

$$|x_n - a| < \frac{1}{n} \quad \text{and} \quad \big|f(x_n) - f(a)\big| \geqslant \varepsilon\,.$$

Then the sequence $\{x_n\}_{n=1}^{\infty} \subseteq I$ and $\lim_{n\to\infty} x_n = a$ by the Squeeze Theorem, but $\lim_{n\to\infty} f(x_n) \neq f(a)$ $\big($since $\big|f(x_n) - f(a)\big| \geqslant \varepsilon$ for all $n \in \mathbb{N}\big)$, a contradiction. ▫

Problem 6. Let $A \subseteq B \subseteq \mathbb{R}$ and $A \neq \varnothing$.

(a) (5%) Show that if $\sup A$ exists, then it is unique.

(b) (10%) Show that if $A \subseteq B$ and $\sup B$ exists, then $\sup A$ exists and $\sup A \leqslant \sup B$.

Proof. (a) Suppose that b_1, b_2 are both the least upper bounds of A. We note that a least upper bound is also an upper bound. Therefore, by the fact that b_1 is a least upper bound of A and b_2 is an upper bound of A, we find that $b_1 \leqslant b_2$. Similarly, $b_2 \leqslant b_1$; thus $b_1 = b_2$.

(b) Suppose that $\sup B$ exists. Since $\sup B$ is an upper bound of B, $x \leqslant \sup B$ for all $x \in B$. Since $A \subseteq B$, $x \in B$ as long as $x \in A$; thus if $x \in A$, $x \leqslant \sup B$. In other words, $\sup B$ is also an upper bound of A or equivalently, A is bounded from above. By the fact that $\mathbb{R}$ is complete, $\sup A$ exists. Since $\sup A$ is the least upper bound of A, we must have $\sup A \leqslant \sup B$. ▫