§5.3 Countable Sets

Theorem

Let S be a non-empty set. The following statements are equivalent:

1. *S is countable;*
2. *there exists a surjection $f : \mathbb{N} \to S$;*
3. *there exists an injection $f : S \to \mathbb{N}$.*

Proof.

"①⇒②" First suppose that $S = \{x_1, \cdots, x_n\}$ is finite. Define $f : \mathbb{N} \to S$ by

$$f(k) = \begin{cases} x_k & \text{if } k < n\,, \\ x_n & \text{if } k \geqslant n\,. \end{cases}$$

Then $f : \mathbb{N} \to S$ is a surjection. Now suppose that S is denumerable. Then by definition of countability, there exists $f : \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S$. □

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1. S is countable;
2. there exists a surjection $f : \mathbb{N} \to S$;

Proof. (Cont'd).

"① ⇐ ②" W.L.O.G. we assume that S is an infinite set. Let $k_1 = 1$. Since $\#(S) = \infty$, $S_1 \equiv S - \{f(k_1)\} \neq \varnothing$; thus $N_1 \equiv f^{-1}(S_1)$ is a non-empty subset of $\mathbb{N}$. By the well-ordered principle (**WOP**) of $\mathbb{N}$, N_1 has a smallest element denoted by k_2. Since $\#(S) = \infty$, $S_2 = S - \{f(k_1), f(k_2)\} \neq \varnothing$; thus $N_2 \equiv f^{-1}(S_2)$ is a non-empty subset of $\mathbb{N}$ and possesses a smallest element denoted by k_3. We continue this process and obtain a set $\{k_1, k_2, \cdots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \cdots$, and k_j is the smallest element of $N_{j-1} \equiv f^{-1}(S - \{f(k_1), f(k_2), \cdots, f(k_{j-1})\})$. □

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Proof. (Cont'd).

Claim: $f\colon \{k_1, k_2, \cdots\} \to S$ is one-to-one and onto.

Proof of claim: The injectivity of f is easy to see since $f(k_j) \notin \{f(k_1), f(k_2), \cdots, f(k_{j-1})\}$ for all $j \geqslant 2$. For surjectivity, assume the contrary that there is $s \in S$ such that $s \notin f(\{k_1, k_2, \cdots\})$. Since $f\colon \mathbb{N} \to \mathbb{S}$ is onto, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$; thus possesses a smallest element k. Since $s \notin f(\{k_1, k_2, \cdots\})$, there exists $\ell \in \mathbb{N}$ such that $k_\ell < k < k_{\ell+1}$. Therefore, $k \in N_\ell$ and $k < k_{\ell+1}$ which contradicts to the fact that $k_{\ell+1}$ is the smallest element of N_ℓ. □

Let $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ be defined by $g(j) = k_j$. Then g is one-to-one and onto; thus $h = g \circ f\colon \mathbb{N} \xrightarrow[\textit{onto}]{1-1} S$. □

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1. S is countable;
3. there exists an injection $f: S \to \mathbb{N}$.

Proof. (Cont'd).

"① ⇒ ③" If $S = \{x_1, \cdots, x_n\}$ is finite, we simply let $f: S \to \mathbb{N}$ be $f(x_n) = n$. Then f is clearly an injection. If S is denumerable, by definition there exists $g : \mathbb{N} \xrightarrow[onto]{1-1} S$ which implies that $f = g^{-1} : S \to \mathbb{N}$ is an injection. □

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1. S is countable;
3. there exists an injection $f : S \to \mathbb{N}$.

Proof. (Cont'd).

"① ⇐ ③" Let $f : S \to \mathbb{N}$ be an injection. If f is also surjective, then $f : S \xrightarrow[\text{onto}]{1-1} \mathbb{N}$ which implies that S is denumerable. Now suppose that $f(S) \subsetneq \mathbb{N}$. Since S is non-empty, there exists $s \in S$. Let $g : \mathbb{N} \to S$ be defined by

$$g(n) = \begin{cases} f^{-1}(n) & \text{if } n \in f(S)\,, \\ s & \text{if } n \notin f(S)\,. \end{cases}$$

Then clearly $g : \mathbb{N} \to S$ is surjective; thus the equivalence between ① and ② implies that S is countable. □

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Example

We have seen that the set $\mathbb{N} \times \mathbb{N}$ is countable. Now consider the map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(m, n) = 2^m 3^n$. This map is not a bijection; however, it is an injection; thus the theorem above implies that $\mathbb{N} \times \mathbb{N}$ is countable.

Example

The set $\mathbb{Q}^+$ of positive rational numbers is denumerable. Since $\mathbb{Q}^+$ is infinite, it suffice to check the countability of $\mathbb{Q}^+$. Consider the map $f : \mathbb{N}^2 \to \mathbb{Q}^+$ defined by $f(m, n) = \frac{m}{n}$. Then f is onto $\mathbb{Q}^+$; thus the theorem above implies that $\mathbb{Q}^+$ is countable.

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Theorem

Any non-empty subset of a countable set is countable.

Proof.

Let S be a countable set, and A be a non-empty subset of S. Since S is countable, by the previous theorem there exists a surjection $f : \mathbb{N} \to S$. On the other hand, since A is a non-empty subset of S, there exists $a \in A$. Define

$$g(x) = \begin{cases} x & \text{if } x \in A\,, \\ a & \text{if } x \notin A\,. \end{cases}$$

Then $g : S \to A$ is a surjection; thus $h = g \circ f : \mathbb{N} \to A$ is also a surjection. The previous theorem shows that A is countable. □

Corollary

A set A is countable if and only if $A \approx S$ for some $S \subseteq \mathbb{N}$.