Chapter 4. Functions

§4.1 Functions as Relations

Recall the usual definition of functions from A to B:

Definition

Let A and B be sets. A ***function*** $f : A \to B$ consists of two sets A and B together with a "rule" that assigns to each $x \in A$ a special element of B denoted by $f(x)$. One writes $x \mapsto f(x)$ to denote that x is mapped to the element $f(x)$. A is called the ***domain*** of f, and B is called the ***target*** or ***co-domain*** of f. The ***range*** of f or the ***image*** of f, is the subset of B defined by $f(A) = \{f(x) \mid x \in A\}$.

Each function is associated with a collection of ordered pairs

$$\{(x, f(x)) \mid x \in A\} \subseteq A \times B\,.$$

Since a collection of ordered pairs is a relation, we can say that a function is a relation from one set to another.

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However, not every relation can serve as a function. A function is a relation with additional special properties and we have the following

Definition (Alternative Definition of Functions)

A ***function*** (or ***mapping***) from A to B is a relation f from A to B such that

1. the domain of f is A; that is, $(\forall x \in A)(\exists y \in B)((x, y) \in f)$, and
2. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

We write $f : A \to B$, and this is read "f is a function from A to B" or "f maps A to B". The set B is called the ***co-domain*** of f. In the case where $B = A$, we say f is a function on A.

When $(x, y) \in f$, we write $y = f(x)$ instead of xfy. We say that y is the ***image*** of f at x (or value of f at x) and that x is a ***pre-image*** of y.

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Remark:

1. A function has only one domain and one range but many possible co-domains.
2. A function on $\mathbb{R}$ is usually called a real-valued function or simply real function. The domain of a real function is usually understood to be the largest possible subset of $\mathbb{R}$ on which the function takes values.

Definition

A function x with domain $\mathbb{N}$ is called an ***infinite sequence***, or simply a ***sequence***. The image of the natural number n is usually written as x_n instead of $x(n)$ and is called the ***n-th term of the sequence***.

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Definition

Let A, B be sets, and $A \subseteq B$.

1. The the ***identity function/map*** on A is the function $I_A : A \to A$ given by $I_A(x) = x$ for all $x \in A$.
2. The ***inclusion function/map*** from A to B is the function $\iota : A \to B$ given by $\iota(x) = x$ for all $x \in A$.
3. The ***characteristic/indicator function*** of A (defined on B) is the map $\mathbf{1}_A : B \to \mathbb{R}$ given by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A\,, \\ 0 & \text{if } x \in B \backslash A\,. \end{cases}$$

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Definition (Cont'd)

4. The ***greatest integer function*** on $\mathbb{R}$ is the function $[\cdot] : \mathbb{R} \to \mathbb{Z}$ given by

$$[x] = \text{the largest integer which is not greater than } x .$$

The function $[\cdot]$ is also called the ***floor function*** or the ***Gauss function***.

5. Let R be an equivalence relation on A. The ***canonical map*** for the equivalence relation R is the map from A to A/R which maps $a \in A$ to $\overline{x}$, the equivalence class of a modulo R.

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Theorem

Two functions f and g are equal if and only if

1. $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, *and*
2. *for all* $x \in \mathrm{Dom}(f)$, $f(x) = g(x)$.

Example

The identity map of A and the inclusion map from A to B are identical functions.

Example

$f(x) = \frac{x}{x}$ and $g(x) = 1$ are different functions since they have different domains.

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Remark:
When a rule of correspondence assigns more than one values to an object in the domain, we say "the function is not well-defined", meaning that it is not really a function. A proof that a function is well-defined is nothing more than a proof that the relation defined by a given rule is single valued.

Example

Let $\bar{x}$ denote the equivalence class of x modulo the congruence relation modulo 4 and $\tilde{y}$ denote the equivalence class of y modulo the congruence relation modulo 10. Define $f(\bar{x}) = \widetilde{2 \cdot x}$. Then this "function" is not really a function since $\bar{0} = \bar{4}$ but $\widetilde{2 \cdot 0} = \tilde{0}$ while $\widetilde{2 \cdot 4} = \tilde{8} \neq \tilde{0}$. In other words, the way f assigns value to $\bar{x}$ is not well-defined.

§4.1 Functions as Relations

Example

Let $\bar{x}$ denote the equivalence class of x modulo the congruence relation modulo 8 and $\tilde{y}$ denote the equivalence class of y modulo the congruence relation modulo 4. The function $f : \mathbb{Z}_8 \to \mathbb{Z}_4$ given by $f(\bar{x}) = \widetilde{x+2}$ is well-defined. To see this, suppose that $\bar{x} = \bar{z}$ in $\mathbb{Z}_8$. Then 8 divides $x - z$ which implies that 4 divides $x - z$; thus 4 divides $(x+2)-(z+2)$. Therefore, $x + 2 = z + 2 \pmod 4$ or equivalently, $\widetilde{x+2} = \widetilde{z+2}$. So f is well-defined.

§4.2 Construction of Functions

Definition

Let $f: A \to B$. The ***inverse*** of f is the relation from B to A:

$$f^{-1} = \{(y,x) \in B \times A \,|\, y = f(x)\} = \{(y,x) \in B \times A \,|\, (x,y) \in f\}.$$

When f^{-1} describes a function, f^{-1} is called the ***inverse function/ map*** of f.

Definition

Let $f: A \to B$ and $g: B \to C$ be functions. The ***composite*** of f and g is the relation from A to C:

$$g \circ f = \{(x,z) \in A \times C \,|\, \text{there exists (a unique) } y \in B \text{ such that } (x,y) \in f \text{ and } (y,z) \in g\}.$$

§4.2 Construction of Functions

Remark: Using the notation in the definition of functions, if $(x, z) \in g \circ f$, then $z = (g \circ f)(x)$. On the other hand, if $(x, z) \in g \circ f$, there exists (a unique) $y \in B$ such that $(x, y) \in f$ and $(y, z) \in g$. Then $y = f(x)$ and $z = g(y)$. Therefore, we also have $z = g(f(x))$; thus $(g \circ f)(x) = g(f(x))$.

Theorem

Let A, B and C be sets, and $f : A \to B$ and $g : B \to C$ be functions. Then $g \circ f$ is a function from A to C.

§4.2 Construction of Functions

Proof of $g \circ f$ is a function from A to C.

By the definition of composition of relations, $g \circ f$ is a relation from A to C.

1. First, we show that $\text{Dom}(g \circ f) = A$. Clearly $\text{Dom}(g \circ f) \subseteq A$, so it suffices to show that $A \subseteq \text{Dom}(g \circ f)$. Let $x \in A$. Since $f : A \to B$ is a function, there exists $y \in B$ such that $(x, y) \in f$. Since $g : B \to C$ is a function, there exists $z \in C$ such that $(y, z) \in g$. This shows that for every $x \in A$, there exists $z \in C$ such that $(x, z) \in g \circ f$; thus $\text{Dom}(g \circ f) = A$.
2. Next, we show that if $(x, z_1) \in g \circ f$ and $(x, z_2) \in g \circ f$, then $z_1 = z_2$. Suppose that $(x, z_1) \in g \circ f$ and $(x, z_2) \in g \circ f$. Then there exists $y_1, y_2 \in B$ such that $(x, y_1) \in f$ and $(y_1, z_1) \in g$, while $(x, y_2) \in f$ and $(y_2, z_2) \in g$. Since f is a function, $y_1 = y_2$; thus that g is a function implies that $z_1 = z_2$. □

§4.2 Construction of Functions

Recall that if A, B, C, D are sets, R be a relation from A to B, S be a relation from B to C, and T be a relation from C to D. Then

1. $T \circ (S \circ R) = (T \circ S) \circ R$.
2. $I_B \circ R = R$ and $R \circ I_A = R$.

Theorem

Let A, B, C, D be sets, and $f : A \to B$, $g : B \to C$, $h : C \to D$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Theorem

Let $f : A \to B$ be a function. Then $f \circ I_A = f$ and $I_B \circ f = f$.

Theorem

Let $f : A \to B$ be a function, and $C = \mathrm{Rng}(f)$. If $f^{-1} : C \to A$ is a function, then $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_C$.

§4.2 Construction of Functions

Definition

Let $f: A \to B$ be a function, and $D \subseteq A$. The ***restriction*** of f to D, denoted by $f|_D$, is the function

$$f|_D = \{(x, y) \mid y = f(x) \text{ and } x \in D\}.$$

If g and h are functions and g is a restriction of h, the h is called an ***extension*** of g.

Example

Let F and G be functions

$$F = \{(1, 2), (2, 6), (3, -9), (5, 7)\},$$
$$G = \{(1, 8), (2, 6), (4, 8), (5, 7), (8, 3)\}.$$

Then $F \cap G = \{(2, 6), (5, 7)\}$ is a function with domain $\{2, 5\}$ which is a proper subset of $\mathrm{Dom}(F) \cap \mathrm{Dom}(G) = \{1, 2, 5\}$.
On the other hand, $\{(1, 2), (1, 8)\} \subseteq F \cup G$; thus $F \cup G$ cannot be a function.

§4.2 Construction of Functions

Theorem

Suppose that f and g are functions. Then $f \cap g$ is a function with domain $A = \{x \mid f(x) = g(x)\}$, and $f \cap g = f|_A = g|_A$.

Proof.

Let $(x, y) \in f \cap g$. Then $y = f(x) = g(x)$; thus

$$\mathrm{Dom}(f \cap g) = \{x \mid f(x) = g(x)\} (\equiv A)\,.$$

If $(x, y_1), (x, y_2) \in f \cap g$, $(x, y_1), (x, y_2) \in f$ which, by the fact that f is a function, implies that $y_1 = y_2$. Therefore, $f \cap g$ is a function. Moreover,

$$f \cap g = \{(x, y) \mid \exists\, x \in A, y = f(x)\}$$

which implies that $f \cap g = f|_A$. □