§3.2 Equivalence Relations

Definition

Let A be a set and R be a relation on A.

1. R is ***reflexive*** on A if $(\forall\, x \in A)(xRx)$.
2. R is ***symmetric*** on A if $\big[\forall\,(x,y) \in A \times A\big](xRy \Leftrightarrow yRx)$.
3. R is ***transitive*** on A if

$$\big[\forall\,(x,y,z) \in A \times A \times A\big]\Big[\big[(xRy) \wedge (yRz)\big] \Rightarrow (xRz)\Big].$$

A relation R on A which is reflexive, symmetric and transitive is called an ***equivalence relation*** on A.

An equivalence relation is often denoted by $\sim$ (the same symbol as negation but $\sim$ as negation is always in front of a proposition while $\sim$ as an equivalence relation is always between two elements in a set).

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Example

The relation "divides" on $\mathbb{N}$ is reflexive and transitive, but not symmetric. The relation "is greater than" on $\mathbb{N}$ is only transitive（遞移律）but not reflexive and transitive.

Example

Let A be a set. The relation "is a subset of" on the power set $\mathcal{P}(A)$ is reflexive, transitive but not symmetric.

Example

The relation $S = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \,\middle|\, x^2 = y^2\right\}$ is reflexive, symmetric and transitive on $\mathbb{R}$.

Example

The relation R on $\mathbb{Z}$ defined by $R = \left\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, x + y \text{ is even}\right\}$ is reflexive, symmetric and transitive.

§3.2 Equivalence Relations

Definition

Let A be a set and R be an equivalence relation on A. For $x \in A$, the ***equivalence class of*** x ***modulo*** R (or simply x ***mod*** R) is a subset of A given by

$$\bar{x} = \{y \in A \mid xRy\}.$$

Each element of $\bar{x}$ is called a ***representative*** of this class. The collection of all equivalence classes modulo R, called A ***modulo*** R, is denoted by A/R (and is the set $A/R = \{\bar{x} \mid x \in A\}$).

Example

The relation $H = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ is an equivalence relation on the set $A = \{1,2,3\}$. Then

$$\bar{1} = \bar{2} = \{1,2\} \quad \text{and} \quad \bar{3} = \{3\}.$$

Therefore, $A/H = \{\{1,2\},\{3\}\}$.