§2.5 Equivalent Forms of Induction

Theorem (Division Algorithm)

For all integers a and b, where $a \neq 0$, there exist a unique pair of integers (q, r) such that $b = aq + r$ and $0 \leqslant r < |a|$. In notation,

$$(\forall\, (a, b) \in (\mathbb{Z}\backslash\{0\}) \times \mathbb{Z})(\exists!(q, r) \in \mathbb{Z} \times \mathbb{Z})\big[(a = bq + r) \wedge (0 \leqslant r < |a|)\big] .$$

Proof.

W.L.O.G., we assume that $a > 0$ and a does not divide b. Define

$$S = \big\{ b - ak \,\big|\, k \in \mathbb{Z} \text{ and } b - ak \geqslant 0 \big\} .$$

Then $0 \notin S$ (which implies that $b \neq 0$). It is clear that if $b > 0$, then $S \neq \varnothing$. If $b < 0$, then $-b > 0$; thus the Archimedean property implies that there exists $k \in \mathbb{N}$ such that $ak > -b$. Therefore, $b - a(-k) > 0$ which also implies that $S \neq \varnothing$. In either case, S is a non-empty subset of $\mathbb{N}$; thus **WOP** implies that S has a smallest element r. Then $b - aq = r$ for some $q \in \mathbb{Z}$; thus $b = aq + r$ and $r > 0$. □

§2.5 Equivalent Forms of Induction

Proof (Cont'd).

Next, we show that $r < |a| = a$. Assume the contrary that $r \geqslant |a| = a$. Then $b - a(q+1) = b - aq - a = r - a \geqslant 0$. Since we assume that $0 \notin S$, we must have $b - a(q+1) > 0$. Therefore,

$$0 < b - a(q+1) = r - a < r = b - aq$$

which shows that r is not the smallest element of S, a contradiction.

To complete the proof, we need to show that the pair (q, r) is unique. Suppose that there exist (q_1, r_1) and (q_2, r_2), where $0 \leqslant r_1, r_2 < |a|$, such that

$$b = aq_1 + r_1 = aq_2 + r_2 .$$

W.L.O.G., we can assume that $r_1 \geqslant r_2$; thus $a(q_2 - q_1) = r_1 - r_2 \geqslant 0$. Therefore, a divides $r_1 - r_2$ which is impossible if $0 < r_1 - r_2 < a$. Therefore, $r_1 = r_2$ and then $q_1 = q_2$. □

Chapter 3. Relations and Partitions

§3.1 Relations

Definition

Let A and B be sets. R is a ***relation*** from A to B if R is a subset of $A \times B$. A relation from A to A is called a relation on A. If $(a, b) \in R$, we say a is R-related (or simply related) to b and write aRb. If $(a, b) \notin R$, we write $a \not R b$.

Example

Let R be the relation "is older than" on the set of all people. If a is 32 yrs old, b is 25 yrs old, and c is 45 yrs old, then aRb, cRb, $a \not R c$. Similarly, the "less than" relation on $\mathbb{R}$ is the set $\{(x, y) \mid x < y\}$.

§3.1 Relations

Remark:

Let A and B be sets. Every subset of $A \times B$ is a relations from A to B; thus every collection of ordered pairs is a relation. In particular, the empty set $\varnothing$ and the set $A \times B$ are relations from A to B ($R = \varnothing$ is the relation that "nothing" is related, while $R = A \times B$ is the relation that "everything" is related).

§3.1 Relations

Definition

For any set A, the ***identity relation on*** A is the (diagonal) set

$$I_A = \{(a, a) \mid a \in A\} .$$

Definition

Let A and B be sets, and R be a relation from A to B. The ***domain*** of R is the set

$$\mathrm{Dom}(R) = \{x \in A \mid (\exists\, y \in B)(xRy)\} ,$$

and the ***range*** of R is the set

$$\mathrm{Rng}(R) = \{y \in B \mid (\exists\, x \in A)(xRy)\} .$$

In other words, the domain of a relation R from A to B is the collection of all first coordinate of ordered pairs in R, and the range of R is the collection of all second coordinates.

§3.1 Relations

Definition

Let A and B be sets, and R be a relation from A to B. The ***inverse*** of R, denoted by R^{-1}, is the relation

$$R^{-1} = \left\{(y,x) \in B \times A \,\middle|\, (x,y) \in R \text{ (or equivalently, } xRy)\right\}.$$

In other words, xRy if and only if $yR^{-1}x$ or equivalently, $(x,y) \in R$ if and only if $(y,x) \in R^{-1}$.

Example

Let $T = \left\{(x,y) \in \mathbb{R} \times \mathbb{R} \,\middle|\, y < 4x^2 - 7\right\}$. To find the inverse of T, we note that

$$\begin{aligned}(x,y) \in T^{-1} &\Leftrightarrow (y,x) \in T \Leftrightarrow x < 4y^2 - 7 \Leftrightarrow x + 7 < 4y^2 \\ &\Leftrightarrow (x,y) \in \left\{(x,y) \in \mathbb{R} \times \mathbb{R} \,\middle|\, x + 7 < 0\right\} \cup \\ &\qquad \left\{(x,y) \in \mathbb{R} \times \mathbb{R} \,\middle|\, 0 \leqslant \frac{x+7}{4} < y^2\right\}.\end{aligned}$$

§3.1 Relations

Theorem

Let A and B be sets, and R be a relation from A to B.

1. $\mathrm{Dom}(R^{-1}) = \mathrm{Rng}(R)$.
2. $\mathrm{Rng}(R^{-1}) = \mathrm{Dom}(R)$.

Proof.

The theorem is concluded by

$$\begin{aligned} b \in \mathrm{Dom}(R^{-1}) &\Leftrightarrow (\exists\, a \in A)\big[(b,a) \in R^{-1}\big] \Leftrightarrow (\exists\, a \in A)\big[(a,b) \in R\big] \\ &\Leftrightarrow b \in \mathrm{Rng}(R)\,, \end{aligned}$$

and

$$\begin{aligned} a \in \mathrm{Rng}(R^{-1}) &\Leftrightarrow (\exists\, b \in B)\big[(b,a) \in R^{-1}\big] \Leftrightarrow (\exists\, b \in B)\big[(a,b) \in R\big] \\ &\Leftrightarrow a \in \mathrm{Dom}(R)\,. \end{aligned}$$

□

§3.1 Relations

Definition

Let A, B, C be sets, and R be a relation from A to B, S be a relation from B to C. The ***composite*** of R and S is a relation from A to C, denoted by $S \circ R$, given by

$$S \circ R = \Big\{(a, c) \in A \times C \,\Big|\, (\exists\, b \in B)\big[(aRb) \wedge (bSc)\big]\Big\}.$$

We note that $\mathrm{Dom}(S \circ R) \subseteq \mathrm{Dom}(R)$ and it may happen that $\mathrm{Dom}(S \circ R) \subsetneq \mathrm{Dom}(R)$.

§3.1 Relations

Example

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{p, q, r, s, t\}$ and $C = \{x, y, z, w\}$. Let R be the relation from A to B:

$$R = \{(1, p), (1, q), (2, q), (3, r), (4, s)\}$$

and S be the relation from B to C:

$$S = \{(p, x), (q, x), (q, y), (s, z), (t, z)\}.$$

Then $S \circ R = \{(1, x), (1, y), (2, x), (2, y), (4, z)\}$.

Example

Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x + 1\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$. Then

$$R \circ S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 + 1\},$$
$$S \circ R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = (x + 1)^2\}.$$

Therefore, $S \circ R \neq R \circ S$.

§3.1 Relations

Theorem

Suppose that A, B, C, D are sets, R be a relation from A to B, S be a relation from B to C, and T be a relation from C to D.

(a) $(R^{-1})^{-1} = R$.

(b) $T \circ (S \circ R) = (T \circ S) \circ R$ (*so composition is associative*).

(c) $I_B \circ R = R$ *and* $R \circ I_A = R$.

(d) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof of (a).

(a) holds since

$$(a, b) \in (R^{-1})^{-1} \Leftrightarrow (b, a) \in R^{-1} \Leftrightarrow (a, b) \in R.$$

□

§3.1 Relations

Proof of (b) $T \circ (S \circ R) = (T \circ S) \circ R$.

Since $S \circ R$ is a relation from A to C, $T \circ (S \circ R)$ is a relation from $A \to D$. Similarly, $(T \circ S) \circ R$ is also a relation from A to D. Let $(a, d) \in A \times D$. Then

$$\begin{aligned}
&(a, d) \in T \circ (S \circ R) \\
&\quad \Leftrightarrow (\exists\, c \in C)\big[(a, c) \in S \circ R \wedge (c, d) \in T\big] \\
&\quad \Leftrightarrow (\exists\, c \in C)(\exists\, b \in B)\big[(a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T\big] \\
&\quad \Leftrightarrow (\exists\, (b, c) \in B \times C)\big[(a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T\big] \\
&\quad \Leftrightarrow (\exists\, b \in B)(\exists\, c \in C)\big[(a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T\big] \\
&\quad \Leftrightarrow (\exists\, b \in B)\big[(a, b) \in R \wedge (b, d) \in T \circ S\big] \\
&\quad \Leftrightarrow (a, d) \in (T \circ S) \circ R\,.
\end{aligned}$$

Therefore, $T \circ (S \circ R) = (T \circ S) \circ R$. □

§3.1 Relations

Proof of (c) $I_B \circ R = R = R \circ I_A$.

Let $(a, b) \in A \times B$ be given. Then

$$(a, b) \in I_B \circ R \Leftrightarrow (\exists\, c \in B)\big[(a, c) \in R \wedge (c, b) \in I_B\big]\,.$$

Note that $(c, b) \in I_B$ if and only if $c = b$; thus

$$(\exists\, c \in B)\big[(a, c) \in R \wedge (c, b) \in I_B\big] \Leftrightarrow (a, b) \in R\,.$$

Therefore, $(a, b) \in I_B \circ R \Leftrightarrow (a, b) \in R$. Similarly, $(a, b) \in R \circ I_A \Leftrightarrow (a, b) \in R$. □

Proof of (d) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Let $(a, c) \in A \times C$. Then

$$\begin{aligned}(c, a) \in (S \circ R)^{-1} &\Leftrightarrow (a, c) \in S \circ R \\ &\Leftrightarrow (\exists\, b \in B)\big[(a, b) \in R \wedge (b, c) \in S\big] \\ &\Leftrightarrow (\exists\, b \in B)\big[(c, b) \in S^{-1} \wedge (b, a) \in R^{-1}\big] \\ &\Leftrightarrow (c, a) \in R^{-1} \circ S^{-1}\,.\end{aligned}$$

□