基礎數學 MA-1015A

Example

Let $\mathcal{F}$ be the collection of sets given by

$$\mathcal{F} = \left\{ \left[\frac{1}{n}, 2 - \frac{1}{n}\right] \,\middle|\, n \in \mathbb{N} \right\}.$$

Then $\bigcup_{A \in \mathcal{F}} A = (0, 2)$ and $\bigcap_{A \in \mathcal{F}} A = \{1\}$. We also write $\bigcup_{A \in \mathcal{F}} A$ and $\bigcap_{A \in \mathcal{F}} A$ as $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$ and $\bigcap_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$, respectively.

Example

Let $\mathcal{F}$ be the collection of sets given by

$$\mathcal{F} = \left\{ \left(-\frac{1}{n}, 2 + \frac{1}{n}\right) \,\middle|\, n \in \mathbb{N} \right\}.$$

Then $\bigcup_{A \in \mathcal{F}} A = (-1, 3)$ and $\bigcap_{A \in \mathcal{F}} A = [0, 2]$. We also write $\bigcup_{A \in \mathcal{F}} A$ and $\bigcap_{A \in \mathcal{F}} A$ as $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 2 + \frac{1}{n}\right)$ and $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 2 + \frac{1}{n}\right)$, respectively.

Theorem

Let $\mathcal{F}$ be a family of sets.

(a) *For every set B in the family $\mathcal{F}$, $\bigcap_{A\in\mathcal{F}} A \subseteq B$.*

(b) *For every set B in the family $\mathcal{F}$, $B \subseteq \bigcup_{A\in\mathcal{F}} A$.*

(c) *If the family $\mathcal{F}$ is non-empty, then $\bigcap_{A\in\mathcal{F}} A \subseteq \bigcup_{A\in\mathcal{F}} A$.*

(d) $\Big(\bigcap_{A\in\mathcal{F}} A\Big)^{\complement} = \bigcup_{A\in\mathcal{F}} A^{\complement}$.

(e) $\Big(\bigcup_{A\in\mathcal{F}} A\Big)^{\complement} = \bigcap_{A\in\mathcal{F}} A^{\complement}$.

(De Morgan's Law)

Proof of (d) $\left(\bigcap_{A\in\mathcal{F}} A\right)^{\complement} = \bigcup_{A\in\mathcal{F}} A^{\complement}$.

Let x be an element in the universe. Then

$$
\begin{aligned}
x \in \left(\bigcap_{A\in\mathcal{F}} A\right)^{\complement} &\text{ if and only if } x \notin \bigcap_{A\in\mathcal{F}} A \\
&\text{ if and only if } \sim\left(x \in \bigcap_{A\in\mathcal{F}} A\right) \\
&\text{ if and only if } \sim(\forall A \in \mathcal{F})(x \in A) \\
&\text{ if and only if } (\exists A \in \mathcal{F}) \sim (x \in A) \\
&\text{ if and only if } (\exists A \in \mathcal{F})(x \notin A) \\
&\text{ if and only if } (\exists A \in \mathcal{F})(x \in A^{\complement}) \\
&\text{ if and only if } x \in \bigcup_{A\in\mathcal{F}} A^{\complement}.
\end{aligned}
$$

□

Theorem

Let $\mathcal{F}$ be a non-empty family of sets and B a set.

1. *If $B \subseteq A$ for all $A \in \mathcal{F}$, then $B \subseteq \bigcap_{A \in \mathcal{F}} A$.*
2. *If $A \subseteq B$ for all $A \in \mathcal{F}$, then $\bigcup_{A \in \mathcal{F}} A \subseteq B$.*

Proof.

1. Suppose that $B \subseteq A$ for all $A \in \mathcal{F}$, and $x \in B$. Then $x \in A$ for all $A \in \mathcal{F}$. Therefore, $(\forall A \in \mathcal{F})(x \in A)$ or equivalently, $x \in \bigcap_{A \in \mathcal{F}} A$.
2. Suppose that $A \subseteq B$ for all $A \in \mathcal{F}$, and $x \in \bigcup_{A \in \mathcal{F}} A$. Then $x \in A$ for some $A \in \mathcal{F}$. By the fact that $A \subseteq B$, we find that $x \in B$. □

Example

Let $\mathcal{F} = \{[-r, r^2+1) \mid r \in \mathbb{R} \text{ and } r \geqslant 0\}$. Then $\bigcup_{A\in\mathcal{F}} A = \mathbb{R}$ and $\bigcap_{A\in\mathcal{F}} = [0,1)$. $\Big($We also write $\bigcup_{A\in\mathcal{F}} A$ and $\bigcap_{A\in\mathcal{F}} A$ as $\bigcup_{r\geqslant 0}[-r, r^2+1)$ and $\bigcap_{r\geqslant 0}[-r, r^2+1)$, respectively.$\Big)$

Proof.

1. If $x \in \mathbb{R}$, then $x \in [-r, r^2+1)$ with $r = |x|$ since$-|x| \leqslant x \leqslant x^2+1$. Therefore, $\mathbb{R} \subseteq \bigcup_{A\in\mathcal{F}} A$.
2. If $x \in [0,1)$, then $x \in [-r, r^2+1)$ for all $r \geqslant 0$; thus $[0,1) \subseteq \bigcap_{A\in\mathcal{F}} A$. If $x \in \bigcap_{A\in\mathcal{F}} A$, then $x \in [-r, r^2+1)$ for all $r \geqslant 0$; thus $x \geqslant -r$ and $x < r^2+1$ for all $r \geqslant 0$. In particular, $x \geqslant 0$ and $x < 1$.

□

§2.3 Indexed Family of Sets

Definition

Let Δ be a non-empty set such that for each $\alpha \in \Delta$ there is a corresponding set A_α. The family $\{A_\alpha \mid \alpha \in \Delta\}$ is an ***indexed family*** of sets, and Δ is called the ***indexing set*** of this family and each $\alpha \in \Delta$ is called an ***index***.

Remark:

1. The indexing set of an indexed family of sets may be finite or infinite, the member sets need not have the same number of elements, and different indices need not correspond to different sets in the family.
2. If $\mathcal{F} = \{A_\alpha \mid \alpha \in \Delta\}$ is an indexed family of sets, we also write $\bigcup_{A \in \mathcal{F}} A$ as $\bigcup_{\alpha \in \Delta} A_\alpha$ and write $\bigcap_{A \in \mathcal{F}} A$ as $\bigcap_{\alpha \in \Delta} A_\alpha$.

3. Another way for the union and intersection of indexed family of sets whose indexing set is $\mathbb{N}$ is

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n .$$

Also, the union and intersection of sets $A_4, A_5, A_6, \cdots, A_{100}$ can be written as

$$\bigcup_{4\leqslant n\leqslant 100} A_n = \bigcup_{n=4}^{100} A_n \quad \text{and} \quad \bigcap_{4\leqslant n\leqslant 100} A_n = \bigcap_{n=4}^{100} A_n$$

and etc.

Definition

The indexed family $\mathcal{F} = \{A_\alpha \mid \alpha \in \Delta\}$ of sets is said to be ***pairwise disjoint*** if for all $\alpha, \beta \in \Delta$, either $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta = \varnothing$.

- **Peano's Axiom for natural numbers**:

1. 1 is a natural number.
2. Every natural number has a unique successor which is a natural number ($+1$ is defined on natural numbers).
3. No two natural numbers have the same successor ($n+1 = m+1$ implies $n = m$).
4. 1 is not a successor for any natural number (1 is the "smallest" natural number).
5. If a property is possessed by 1 and is possessed by the successor of every natural number that possesses it, then the property is possessed by all natural numbers.（如果某個被自然數 1 所擁有的性質，也被其它擁有這個性質的自然數的下一個自然數所擁有，那麼所有的自然數都會擁有這個性質）

- **Principle of Mathematical Induction (PMI)**:

If $S \subseteq \mathbb{N}$ has the property that

1. $1 \in S$, and
2. $n+1 \in S$ whenever $n \in S$,

then $S = \mathbb{N}$.

Definition

A set S of natural numbers is called ***inductive*** if it has the property that whenever $n \in S$, then $n+1 \in S$.

PMI can be rephrased as "if S is an inductive set and $1 \in S$, then $S = \mathbb{N}$".

- **Inductive definition**: Inductive definition is a way to define some "functions" $f(n)$ for all natural numbers n. It is done by describe the first object $f(1)$, and then the $(n+1)$-th object $f(n+1)$ is defined in terms of the n-th object $f(n)$. We remark that in this way of defining f, **PMI** ensures that the collection of all n for which the corresponding object $f(n)$ is defined is $\mathbb{N}$.

Example

The **factorial** $n!$ can be defined by

1. $1! = 1$;
2. For all $n \in \mathbb{N}$, $(n+1)! = n! \times (n+1)$.

Note: one can extend the definition of the factorial function by defining $0! = 1$.

Example

The notation $\sum_{k=1}^{n} x_k$ can be defined by

1. $\sum_{k=1}^{1} x_k = x_1$;
2. For all $n \in \mathbb{N}$, $\sum_{k=1}^{n+1} x_k = \sum_{k=1}^{n} x_k + x_{n+1}$.

Example

The notation $\prod_{k=1}^{n} x_k$ can be defined by

1. $\prod_{k=1}^{1} x_k = x_1$;
2. For all $n \in \mathbb{N}$, $\prod_{k=1}^{n+1} x_k = \left(\prod_{k=1}^{n} x_k\right) \cdot x_{n+1}$.

PMI can provide a powerful method for proving statements that are true for all natural numbers.

Suppose that $\mathrm{P}(n)$ is an open sentence concerning the natural numbers.

Proof of $(\forall\, n \in \mathbb{N})\mathrm{P}(n)$ **by mathematical induction**

Proof.

Let S denote the truth of P.

(i) **Basis Step**. Show that $1 \in S$.

(ii) **Inductive Step**. Show that S is inductive by showing that if $n \in S$, then $n + 1 \in S$.

Therefore, **PMI** ensures that the truth set of P is $\mathbb{N}$. □