§2.2 Set Operations

Definition

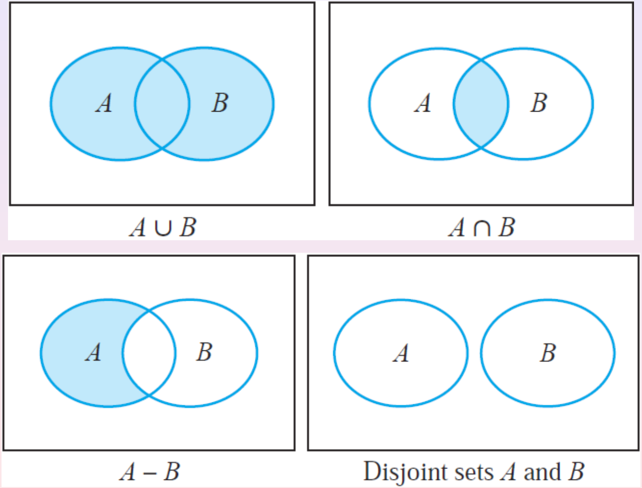
Let A and B be sets.

1. The **union of** A **and** B, denoted by $A \cup B$, is the set
$$\left\{x \middle| (x \in A) \vee (x \in B)\right\}.$$
2. The **intersection of** A **and** B, denoted by $A \cap B$, is the set
$$\left\{x \middle| (x \in A) \wedge (x \in B)\right\}.$$
3. The **difference of** A **and** B, denoted by $A - B$, is the set
$$\left\{x \middle| (x \in A) \wedge (x \notin B)\right\}.$$

Definition

Two sets A and B are said to be ***disjoint*** if $A \cap B = \varnothing$.

- Venn diagrams:



Theorem

Let A, B and C be sets. Then

(a) $A \subseteq A \cup B$; (b) $A \cap B \subseteq A$; (c) $A \cap \varnothing = \varnothing$; (d) $A \cup \varnothing = A$;

(e) $A \cap A = A$; (f) $A \cup A = A$; (g) $A \setminus \varnothing = A$; (h) $\varnothing \setminus A = \varnothing$;

(i) $A \cup B = B \cup A$;
(j) $A \cap B = B \cap A$; } **(commutative laws)**

(k) $A \cup (B \cup C) = (A \cup B) \cup C$;
(ℓ) $A \cap (B \cap C) = (A \cap B) \cap C$; } **(associative laws)**

(m) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(n) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; } **(distributive laws)**

(o) $A \subseteq B \Leftrightarrow A \cup B = B$; (p) $A \subseteq B \Leftrightarrow A \cap B = A$;

(q) $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$; (r) $A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$.

Note: $(A \cup B) \cap C \neq A \cup (B \cap C)$ in general!

Proof of (m) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Let x be an element in the universe, and P, Q and R denote the propositions $x \in A$, $x \in B$ and $x \in C$, respectively. Note that from the truth table, we conclude that

$$\mathrm{P} \wedge (\mathrm{Q} \vee \mathrm{R}) \Leftrightarrow \big[(\mathrm{P} \wedge \mathrm{Q}) \vee (\mathrm{P} \wedge \mathrm{R})\big],$$

1. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$; thus the proposition $\mathrm{P} \wedge (\mathrm{Q} \vee \mathrm{R})$ is true. Therefore, the proposition $\big[(\mathrm{P} \wedge \mathrm{Q}) \vee (\mathrm{P} \wedge \mathrm{R})\big]$ is also true which implies that $x \in A \cap B$ or $x \in A \cap C$; thus
$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$
2. Working conversely, we find that if $x \in A \cap B$ or $x \in A \cap C$, then $x \in A \cap (B \cup C)$. Therefore,
$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

□

Proof of (m) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Thus,

1. if $x \in B$, then $x \in A \cap B$.
2. if $x \in C$, then $x \in A \cap C$.

Therefore, $x \in A \cap B$ or $x \in A \cap C$ which shows $x \in (A \cap B) \cup (A \cap C)$; thus we establish that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) .$$

On the other hand, suppose that $x \in (A \cap B) \cup (A \cap C)$.

1. if $x \in A \cap B$, then $x \in A$ and $x \in B$.
2. if $x \in A \cap C$, then $x \in A$ and $x \in C$.

In either cases, $x \in A$; thus if $x \in (A \cap B) \cup (A \cap C)$, then $x \in A$ but at the same time $x \in B$ or $x \in C$. Thus, $x \in A$ and $x \in B \cup C$ which shows that $x \in A \cap (B \cup C)$. Therefore,

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) .$$

□

Proof of (p) $A \subseteq B \Leftrightarrow A \cap B = A$.

($\Rightarrow$) Suppose that $A \subseteq B$. Let x be an element in A. Then $x \in B$ since $A \subseteq B$; thus $x \in A \cap B$ which implies that $A \subseteq A \cap B$. On the other hand, it is clear that $A \cap B \subseteq A$, so we conclude that $A \cap B = A$.

($\Leftarrow$) Suppose that $A \cap B = A$. Let x be an element in A. Then $x \in A \cap B$ which shows that $x \in B$. Therefore, $A \subseteq B$. □

Definition

Let U be the universe and $A \subseteq U$. The ***complement*** (補集) of A, denoted by $A^{\complement}$, is the set $U - A$.

Theorem

Let U be the universe, and $A, B \subseteq U$. Then

(a) $(A^{\complement})^{\complement} = A$. (b) $A \cup A^{\complement} = U$.

(c) $A \cap A^{\complement} = \varnothing$. (d) $A - B = A \cap B^{\complement}$.

(e) $A \subseteq B$ *if and only if* $B^{\complement} \subseteq A^{\complement}$.

(f) $A \cap B = \varnothing$ *if and only if* $A \subseteq B^{\complement}$

(g) $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$.

(h) $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$.

(De Morgan's Law)

Proof of (a) $(A^{\complement})^{\complement} = A$.

By the definition of the complement, $x \in (A^{\complement})^{\complement}$ if and only if $x \notin A^{\complement}$ if and only if $x \in A$. □

Proof of (e) $A \subseteq B \Leftrightarrow B^{\complement} \subseteq A^{\complement}$.

By the equivalence of $\mathrm{P} \Rightarrow \mathrm{Q}$ and $\sim\mathrm{Q} \Rightarrow \sim\mathrm{P}$, we conclude that

$$(\forall\, x)\big[(x \in A) \Rightarrow (x \in B)\big] \quad \Leftrightarrow \quad (\forall\, x)\big[(x \notin B) \Rightarrow (x \notin A)\big]$$

and the bi-directional statement is identical to that

$$A \subseteq B \Leftrightarrow B^{\complement} \subseteq A^{\complement}.$$

□

Alternative proof of (e) $A \subseteq B \Leftrightarrow B^{\complement} \subseteq A^{\complement}$.

Using (a), it suffices to show that $A \subseteq B \Rightarrow B^{\complement} \subseteq A^{\complement}$. Suppose that $A \subseteq B$, but $B^{\complement} \nsubseteq A^{\complement}$. Then there exists $x \in B^{\complement}$ and $x \in A$; however, by the fact that $A \subseteq B$, x has to belong to B, a contradiction. □

Proof of (g) $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$.

By the equivalence of $\sim(\mathrm{P} \vee \mathrm{Q})$ and $(\sim\mathrm{P}) \wedge (\sim\mathrm{Q})$, we find that

$$(\forall\, x) \sim \big[(x \in A) \vee (x \in B)\big] \quad \Leftrightarrow \quad (\forall\, x)\big[(x \notin A) \wedge (x \notin B)\big]$$

and the bi-directional statement is identical to that

$$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}\,. \qquad \square$$

Alternative proof of (g) $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$.

Let x be an element in the universe.

$x \in (A \cup B)^{\complement}$ if and only if $x \notin A \cup B$
if and only if it is not the case that $x \in A$ or $x \in B$
if and only if $x \notin A$ and $x \notin B$
if and only if $x \in A^{\complement}$ and $x \in B^{\complement}$
if and only if $x \in A^{\complement} \cap B^{\complement}$. $\square$

Definition

An ***ordered pair*** (a, b) is an object formed from two objects a and b, where a is called the ***first coordinate*** and b the ***second coordinate***. Two ordered pairs are equal whenever their corresponding coordinates are the same.
An ***ordered n-tuples*** $(a_1, a_2, \cdots, a_n)$ is an object formed from n objects a_1, a_2, $\cdots$, a_n, where a_j is called the j-th coordinate. Two n-tuples $(a_1, a_2, \cdots, a_n)$, $(c_1, c_2, \cdots, c_n)$ are equal if $a_i = c_i$ for $i \in \{1, 2, \cdots, n\}$.

Definition

Let A and B be sets. The product of A and B, denoted by $A \times B$, is

$$A \times B = \left\{ (a, b) \,\middle|\, a \in A, b \in B \right\} .$$

The product of three or more sets are defined similarly.

Example

Let $A = \{1, 3, 5\}$ and $B = \{\star, \diamond\}$. Then

$$A \times B = \{(1, \star), (3, \star), (5, \star), (1, \diamond), (3, \diamond), (5, \diamond)\}.$$

Theorem

If A, B, C and D are sets, then

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(c) $A \times \varnothing = \varnothing$.

(d) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(e) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

(f) $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

Definition

Let $\mathcal{F}$ be a family of sets.

1. The ***union*** of the family $\mathcal{F}$ or the ***union*** over $\mathcal{F}$, denoted by $\bigcup_{A\in\mathcal{F}} A$, is the set $\{x \mid x \in A \text{ for some } A \in \mathcal{F}\}$. Therefore,
$$x \in \bigcup_{A\in\mathcal{F}} A \quad \text{if and only if} \quad (\exists A \in \mathcal{F})(x \in A).$$
2. The ***intersection*** of the family $\mathcal{F}$ or the ***intersection*** over $\mathcal{F}$, denoted by $\bigcap_{A\in\mathcal{F}} A$, is the set $\{x \mid x \in A \text{ for all } A \in \mathcal{F}\}$. Therefore,
$$x \in \bigcap_{A\in\mathcal{F}} A \quad \text{if and only if} \quad (\forall A \in \mathcal{F})(x \in A).$$

Example

Let $\mathcal{F}$ be the collection of sets given by

$$\mathcal{F} = \left\{ \left[\frac{1}{n}, 2 - \frac{1}{n} \right] \,\middle|\, n \in \mathbb{N} \right\}.$$

Then $\bigcup_{A \in \mathcal{F}} A = (0, 2)$ and $\bigcap_{A \in \mathcal{F}} A = \{1\}$. We also write $\bigcup_{A \in \mathcal{F}} A$ and $\bigcap_{A \in \mathcal{F}} A$ as $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n} \right]$ and $\bigcap_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n} \right]$, respectively.

Example

Let $\mathcal{F}$ be the collection of sets given by

$$\mathcal{F} = \left\{ \left(-\frac{1}{n}, 2 + \frac{1}{n} \right) \,\middle|\, n \in \mathbb{N} \right\}.$$

Then $\bigcup_{A \in \mathcal{F}} A = (-1, 3)$ and $\bigcap_{A \in \mathcal{F}} A = [0, 2]$. We also write $\bigcup_{A \in \mathcal{F}} A$ and $\bigcap_{A \in \mathcal{F}} A$ as $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 2 + \frac{1}{n} \right)$ and $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 2 + \frac{1}{n} \right)$, respectively.

Theorem

Let $\mathcal{F}$ be a family of sets.

(a) *For every set B in the family $\mathcal{F}$, $\bigcap_{A\in\mathcal{F}} A \subseteq B$.*

(b) *For every set B in the family $\mathcal{F}$, $B \subseteq \bigcup_{A\in\mathcal{F}} A$.*

(c) *If the family $\mathcal{F}$ is non-empty, then $\bigcap_{A\in\mathcal{F}} A \subseteq \bigcup_{A\in\mathcal{F}} A$.*

(d) $\Big(\bigcap_{A\in\mathcal{F}} A\Big)^{\complement} = \bigcup_{A\in\mathcal{F}} A^{\complement}$.

(e) $\Big(\bigcup_{A\in\mathcal{F}} A\Big)^{\complement} = \bigcap_{A\in\mathcal{F}} A^{\complement}$.

(De Morgan's Law) [for (d) and (e)]

Proof of (d) $\left(\bigcap_{A\in\mathcal{F}} A\right)^{\complement} = \bigcup_{A\in\mathcal{F}} A^{\complement}$.

Let x be an element in the universe. Then

$$\begin{aligned}
x \in \left(\bigcap_{A\in\mathcal{F}} A\right)^{\complement} &\text{ if and only if } x \notin \bigcap_{A\in\mathcal{F}} A \\
&\text{ if and only if } \sim\left(x \in \bigcap_{A\in\mathcal{F}} A\right) \\
&\text{ if and only if } \sim(\forall A \in \mathcal{F})(x \in A) \\
&\text{ if and only if } (\exists A \in \mathcal{F}) \sim(x \in A) \\
&\text{ if and only if } (\exists A \in \mathcal{F})(x \notin A) \\
&\text{ if and only if } (\exists A \in \mathcal{F})(x \in A^{\complement}) \\
&\text{ if and only if } x \in \bigcup_{A\in\mathcal{F}} A^{\complement}.
\end{aligned}$$

□

Theorem

Let $\mathcal{F}$ be a non-empty family of sets and B a set.

1. *If $B \subseteq A$ for all $A \in \mathcal{F}$, then $B \subseteq \bigcap_{A \in \mathcal{F}} A$.*
2. *If $A \subseteq B$ for all $A \in \mathcal{F}$, then $\bigcup_{A \in \mathcal{F}} A \subseteq B$.*

Proof.

1. Suppose that $B \subseteq A$ for all $A \in \mathcal{F}$, and $x \in B$. Then $x \in A$ for all $A \in \mathcal{F}$. Therefore, $(\forall A \in \mathcal{F})(x \in A)$ or equivalently, $x \in \bigcap_{A \in \mathcal{F}} A$.
2. Suppose that $A \subseteq B$ for all $A \in \mathcal{F}$, and $x \in \bigcup_{A \in \mathcal{F}} A$. Then $x \in A$ for some $A \in \mathcal{F}$. By the fact that $A \subseteq B$, we find that $x \in B$. □