基礎數學 MA-1015A

§2.1 Basic Concepts of Set Theory

Definition

Two sets A and B are said to be ***equal***, denoted by $A = B$, if $(\forall x)(x \in A \Leftrightarrow x \in B)$; that is $(A \subseteq B) \wedge (B \subseteq A)$. A set B is said to be a ***proper subset*** of a set A, denoted by $B \subsetneq A$, if $B \subseteq A$ but $A \neq B$.

- **Proof of** $A = B$:

Two-part proof of $A = B$

Proof.

(i) Prove that $A \subseteq B$ (by any method.)

(ii) Prove that $B \subseteq A$ (by any method).

Therefore, $A = B$. □

§2.1 Basic Concepts of Set Theory

Theorem

If A and B are sets with no elements, then $A = B$.

Proof.

Let A, B be set. If A has no element, then $A = \varnothing$; thus by the fact that empty set is a subset of any set, $A \subseteq B$. Similarly, if B has no element, then $B \subseteq A$. □

Theorem

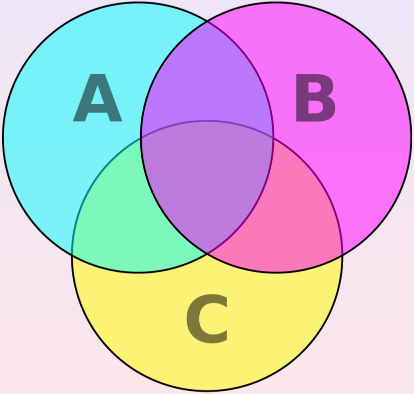
For any sets A and B, if $A \subseteq B$ and $A \neq \varnothing$, then $B \neq \varnothing$.

Proof.

Let A, B be sets, $A \subseteq B$, and $A \neq \varnothing$. Then there is an element x such that $x \in A$. By the assumption that $A \subseteq B$, we must have $x \in B$. Therefore, $B \neq \varnothing$. □

§2.1 Basic Concepts of Set Theory

- **Venn diagrams**:



§2.1 Basic Concepts of Set Theory

Definition

Let A be a set. The ***power set*** of A, denoted by $\mathcal{P}(A)$ or 2^A, is the colloection of all subsets of A. In other words, $\mathcal{P}(A) \equiv \{B \,|\, B \subseteq A\}$.

Example

If $A = \{a, b, c, d\}$, then

$$\mathcal{P}(A) = \Big\{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\Big\}.$$

We note that $\#(A) = 4$ and $\#(\mathcal{P}(A)) = 16 = 2^{\#(A)}$.

§2.1 Basic Concepts of Set Theory

Theorem

If A is a set with n elements, then $\mathcal{P}(A)$ is a set with 2^n elements.

Proof.

Suppose that A is a set with n elements.

1. If $n = 0$, then $A = \varnothing$; thus $\mathcal{P}(A) = \{\varnothing\}$ which shows that $\mathcal{P}(A)$ has $2^0 = 1$ element.
2. If $n \geqslant 1$, we write A as $\{x_1, x_2, \cdots, x_n\}$. To describe a subset B of A, we need to know for each $1 \leqslant i \leqslant n$ whether x_i is in B. For each x_i, there are two possibilities (either $x_i \in B$ or $x_i \notin B$). Thus, there are exactly 2^n different ways of making a subset of A. Therefore, $\mathcal{P}(A)$ has 2^n elements. □

§2.1 Basic Concepts of Set Theory

Theorem

Let A, B be sets. Then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof.

Let A, B be sets.

($\Rightarrow$) Suppose that $A \subseteq B$ and $C \in \mathcal{P}(A)$. Then C is a subset of A; thus the fact that $A \subseteq B$ implies that $C \subseteq B$. Therefore, $C \in \mathcal{P}(B)$.

($\Leftarrow$) Suppose that $A \nsubseteq B$. Then there exists $x \in A$ but $x \notin B$. Then $\{x\} \subseteq A$ but $\{x\} \nsubseteq B$ which shows that $\mathcal{P}(A) \nsubseteq \mathcal{P}(B)$. □

§2.2 Set Operations

Definition

Let A and B be sets.

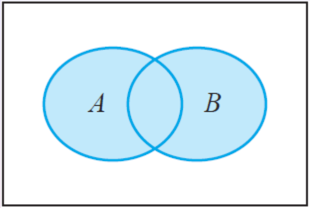
1. The **union of** A **and** B, denoted by $A \cup B$, is the set
$$\left\{x \,\middle|\, (x \in A) \vee (x \in B)\right\}.$$
2. The **intersection of** A **and** B, denoted by $A \cap B$, is the set
$$\left\{x \,\middle|\, (x \in A) \wedge (x \in B)\right\}.$$
3. The **difference of** A **and** B, denoted by $A - B$, is the set
$$\left\{x \,\middle|\, (x \in A) \wedge (x \notin B)\right\}.$$

Definition

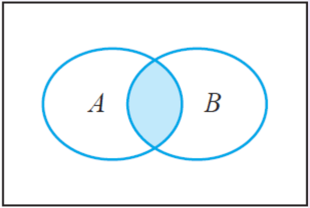
Two sets A and B are said to be ***disjoint*** if $A \cap B = \varnothing$.

§2.2 Set Operations

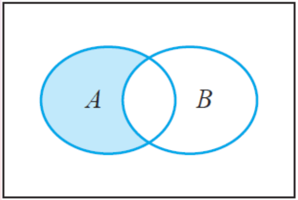
- Venn diagrams:



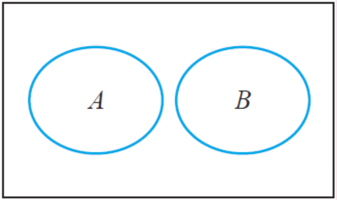
$A \cup B$



$A \cap B$



$A - B$



Disjoint sets A and B

§2.2 Set Operations

Theorem

Let A, B and C be sets. Then

(a) $A \subseteq A \cup B$; (b) $A \cap B \subseteq A$; (c) $A \cap \varnothing = \varnothing$; (d) $A \cup \varnothing = A$;

(e) $A \cap A = A$; (f) $A \cup A = A$; (g) $A \setminus \varnothing = A$; (h) $\varnothing \setminus A = \varnothing$;

(i) $A \cup B = B \cup A$;
(j) $A \cap B = B \cap A$; } **(commutative laws)**

(k) $A \cup (B \cup C) = (A \cup B) \cup C$;
(ℓ) $A \cap (B \cap C) = (A \cap B) \cap C$; } **(associative laws)**

(m) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(n) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; } **(distributive laws)**

(o) $A \subseteq B \Leftrightarrow A \cup B = B$; (p) $A \subseteq B \Leftrightarrow A \cap B = A$;

(q) $A \subseteq B \Rightarrow A \cup C = B \cup C$; (r) $A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$.

Note: $(A \cup B) \cap C \neq A \cup (B \cap C)$ in general!