基礎數學 MA-1015A

Chapter 1. Logic and Proofs

§1.1 Propositions and Connectives

Definition

A ***proposition*** is a sentence that has exactly one truth value. It is either true, which we denote by T, or false, which we denote by F.

Example

$7^2 > 60$ (F), $\pi > 3$ (T), Earth is the closest planet to the sun (F).

Example

The statement "the north Pacific right whale（露脊鯨）will be extinct species before the year 2525" has one truth value but it takes time to determine the truth value.

Example

That "Euclid was left-handed" is a statement that has one truth value but may never be known.

§1.1 Propositions and Connectives

Definition

A ***negation*** of a proposition P, denoted by $\sim\mathrm{P}$, is the proposition "not P". The proposition $\sim\mathrm{P}$ is true / false exactly when P is false / true.

Definition

Given propositions P and Q, the ***conjunction*** / ***disjunction*** of P and Q, denoted by $\mathrm{P} \wedge \mathrm{Q}$ / $\mathrm{P} \vee \mathrm{Q}$, is the proposition "$\mathrm{P}$ and / or Q". $\mathrm{P} \wedge \mathrm{Q}$ / $\mathrm{P} \vee \mathrm{Q}$ is true exactly when both P and Q are true / at least one of P or Q is true.

§1.1 Propositions and Connectives

Example

Now we analyze the sentence "either 7 is prime and 9 is even, or else 11 is not less than 3". Let P denote the sentence "7 is a prime", Q denote the sentence "9 is even", and R denote the sentence "11 is less than 3". Then the original sentence can be symbolized by $(P \wedge Q) \vee (\sim R)$, and the table of truth value for this sentence is

P	Q	R	$P \wedge Q$	$\sim R$	$(P \wedge Q) \vee (\sim R)$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
T	F	F	F	T	T
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Since P is true and Q, R are false, the sentence $(P \wedge Q) \vee (\sim R)$ is true.

§1.1 Propositions and Connectives

Definition

A ***tautology*** / ***contradiction*** is a propositional form that is true / false for every assignment of truth values to its component.

Example

The logic symbol $(\mathrm{P} \vee \mathrm{Q}) \vee (\sim \mathrm{P} \wedge \sim \mathrm{Q})$ is a tautology.

Example

The logic symbol $\sim (\mathrm{P} \vee \sim \mathrm{P}) \vee (\mathrm{Q} \wedge \sim \mathrm{Q})$ is a contradiction.

Definition

Two propositional forms are said to be ***equivalent*** if they have the same truth value.

§1.1 Propositions and Connectives

Theorem

For propositions P, Q, R, *we have the following:*

(a) $P \Leftrightarrow \sim(\sim P)$. (**Double Negation Law**)

(b) $P \vee Q \Leftrightarrow Q \vee P$
(c) $P \wedge Q \Leftrightarrow Q \wedge P$
(**Commutative Laws**)

(d) $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$
(e) $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$
(**Associative Laws**)

(f) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
(g) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
(**Distributive Laws**)

(h) $\sim(P \wedge Q) \Leftrightarrow (\sim P) \vee (\sim Q)$
(i) $\sim(P \vee Q) \Leftrightarrow (\sim P) \wedge (\sim Q)$
(**De Morgan's Laws**)

§1.1 Propositions and Connectives

Proof.

We prove (g) for example, and the other cases can be shown in a similar fashion. Using the truth table,

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

we find that "$P \vee (Q \wedge R)$" is equivalent to "$(P \vee Q) \wedge (P \vee R)$". □

§1.1 Propositions and Connectives

Definition

A ***denial*** of a proposition is any proposition equivalent to $\sim \mathrm{P}$.

- **Rules for $\sim$, $\wedge$ and $\vee$:**
  1. $\sim$ is always applied to the smallest proposition following it.
  2. $\wedge$ connects the smallest propositions surrounding it.
  3. $\vee$ connects the smallest propositions surrounding it.

Example

Under the convention above, we have

1. $\sim \mathrm{P} \vee \sim \mathrm{Q} \Leftrightarrow (\sim \mathrm{P}) \vee (\sim \mathrm{Q})$.
2. $\mathrm{P} \vee \mathrm{Q} \vee \mathrm{R} \Leftrightarrow (\mathrm{P} \vee \mathrm{Q}) \vee \mathrm{R} \Leftrightarrow \mathrm{P} \vee (\mathrm{Q} \vee \mathrm{R})$.
3. $\mathrm{P} \wedge \sim \mathrm{Q} \vee \sim \mathrm{R} \Leftrightarrow \big[\mathrm{P} \wedge (\sim \mathrm{Q})\big] \vee (\sim \mathrm{R})$.
4. $\mathrm{R} \wedge \mathrm{P} \wedge \mathrm{S} \wedge \mathrm{Q} \Leftrightarrow \big[(\mathrm{R} \wedge \mathrm{P}) \wedge \mathrm{S}\big] \wedge \mathrm{Q}$.

§1.2 Conditionals and Biconditionals

Definition

For propositions P and Q, the ***conditional sentence*** $P \Rightarrow Q$ is the proposition "if P, then Q". Proposition P is called the ***antecedent*** and Q is the ***consequence***. The sentence $P \Rightarrow Q$ is true if and only if P is false or Q is true.

Remark:

In a conditional sentence, P and Q might not have connections. The truth value of the sentence "$P \Rightarrow Q$" only depends on the truth value of P and Q.

§1.2 Conditionals and Biconditionals

Example

We would like to determine the truth value of the sentence "if $x > 8$, then $x > 5$". Let P denote the sentence "$x > 8$" and Q the sentence "$x > 5$".

1. If P, Q are both true statements, then $x > 8$ which is (exactly the same as P thus) true.
2. If P is false while Q is true, then $5 < x \leqslant 8$ which is (exactly the same as $\sim\mathrm{P} \wedge \mathrm{Q}$ thus) true.
3. If P, Q are both false statements, then $x \leqslant 5$ which is (exactly the same as $\sim\mathrm{Q}$ thus) true.
4. It is not possible to have P true but Q false.

§1.2 Conditionals and Biconditionals

- **How to read** $P \Rightarrow Q$ **in English?**

1. If P, then Q.
2. P is sufficient for Q.
3. P only if Q.
4. Q whenever P.
5. Q is necessary for P.
6. Q, if/when P.

Definition

Let P and Q be propositions.

1. The ***converse*** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
2. The ***contrapositive*** of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.

§1.2 Conditionals and Biconditionals

Example

We would like to determine the truth value, as well as the converse and the contrapositive, of the sentence "if π is an integer, then 14 is even".

1. Since that π is an integer is false, the implication "if π is an integer, then 14 is even" is true.
2. The converse of the sentence is "if 14 is even, then π is an integer" which is a false statement.
3. The contrapositive of the sentence is "if 14 is not even, then π is not an integer" which is a true statement since the antecedent "14 is not even" is false.

By this example, we know that a sentence and its converse cannot be equivalent.

§1.2 Conditionals and Biconditionals

Theorem

For propositions P *and* Q, *the sentence* $\mathrm{P} \Rightarrow \mathrm{Q}$ *is equivalent to its contrapositive* $\sim\mathrm{Q} \Rightarrow \sim\mathrm{P}$.

Proof.

Using the truth table

P	Q	$\mathrm{P} \Rightarrow \mathrm{Q}$	$\sim\mathrm{Q}$	$\sim\mathrm{P}$	$\sim\mathrm{Q} \Rightarrow \sim\mathrm{P}$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

we conclude that the truth value of $\mathrm{P} \Rightarrow \mathrm{Q}$ and $\sim\mathrm{Q} \Rightarrow \sim\mathrm{P}$ are the same; thus they are equivalent sentences. □

§1.2 Conditionals and Biconditionals

Definition

For propositions P and Q, the ***bi-conditional sentence*** $\mathrm{P} \Leftrightarrow \mathrm{Q}$ is the proposition "P if and only if Q". The sentence $\mathrm{P} \Leftrightarrow \mathrm{Q}$ is true exactly when P and Q have the same truth values. In other words, $\mathrm{P} \Leftrightarrow \mathrm{Q}$ is true if and only if P is equivalent to Q.

Remark: The notation $\Leftrightarrow$ is a combination of $\Rightarrow$ and its converse $\Leftarrow$, so the notation seems to suggest that $(\mathrm{P} \Leftrightarrow \mathrm{Q})$ is equivalent to $(\mathrm{P} \Rightarrow \mathrm{Q}) \wedge (\mathrm{Q} \Rightarrow \mathrm{P})$. This is in fact true since

P	Q	$\mathrm{P} \Leftrightarrow \mathrm{Q}$	$\mathrm{P} \Rightarrow \mathrm{Q}$	$\mathrm{Q} \Rightarrow \mathrm{P}$	$(\mathrm{P} \Rightarrow \mathrm{Q}) \wedge (\mathrm{P} \Rightarrow \mathrm{Q})$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

§1.2 Conditionals and Biconditionals

Example

1. The proposition "$2^3 = 8$ if and only if 49 is a perfect square" is true because both components are true.
2. The proposition "$\pi = \dfrac{22}{7}$ if and only if $\sqrt{2}$ is a rational number" is also true (since both components are false).
3. The proposition "$6 + 1 = 7$ if and only if Argentina is north of the equator" is false because the truth values of the components differ.

§1.2 Conditionals and Biconditionals

Remark:

Definitions may be stated with the "if and only if" wording, but it is also common practice to state a formal definition using the word "if". For example, we could say that "a function f is continuous at a number c if $\cdots$" leaving the "only if" part understood.

Example

A teacher says "If you score 74% or higher on the next test, you will pass the exam". Even though this is a conditional sentence, everyone will interpret the meaning as a biconditional (since the teacher tries to "define" how you can pass the exam).

§1.2 Conditionals and Biconditionals

Theorem

For propositions P, Q *and* R, *we have the following:*

(a) $(\mathrm{P} \Rightarrow \mathrm{Q}) \;\Leftrightarrow\; (\sim\mathrm{P} \vee \mathrm{Q})$.

(b) $(\mathrm{P} \Leftrightarrow \mathrm{Q}) \;\Leftrightarrow\; (\mathrm{P} \Rightarrow \mathrm{Q}) \wedge (\mathrm{Q} \Rightarrow \mathrm{P})$.

(c) $\sim(\mathrm{P} \Rightarrow \mathrm{Q}) \;\Leftrightarrow\; (\mathrm{P} \wedge \sim\mathrm{Q})$.

(d) $\sim(\mathrm{P} \wedge \mathrm{Q}) \;\Leftrightarrow\; (\mathrm{P} \Rightarrow \sim\mathrm{Q})$.

(e) $\sim(\mathrm{P} \wedge \mathrm{Q}) \;\Leftrightarrow\; (\mathrm{Q} \Rightarrow \sim\mathrm{P})$.

(f) $\mathrm{P} \Rightarrow (\mathrm{Q} \Rightarrow \mathrm{R}) \;\Leftrightarrow\; (\mathrm{P} \wedge \mathrm{Q}) \Rightarrow \mathrm{R}$.

(g) $\mathrm{P} \Rightarrow (\mathrm{Q} \wedge \mathrm{R}) \;\Leftrightarrow\; (\mathrm{P} \Rightarrow \mathrm{Q}) \wedge (\mathrm{P} \Rightarrow \mathrm{R})$.

(h) $(\mathrm{P} \vee \mathrm{Q}) \Rightarrow \mathrm{R} \;\Leftrightarrow\; (\mathrm{P} \Rightarrow \mathrm{R}) \wedge (\mathrm{Q} \Rightarrow \mathrm{R})$.

§1.2 Conditionals and Biconditionals

- **How to read** $\mathrm{P} \Leftrightarrow \mathrm{Q}$ **in English**?

1. P if and only if Q.
2. P if, but only if, Q.
3. P implies Q, and conversely.
4. P is equivalent to Q.
5. P is necessary and sufficient for Q.

- **Rules for** $\sim$**,** $\wedge$**,** $\vee$**,** $\Rightarrow$ **and** $\Leftrightarrow$: These connectives are always applied in the order listed.

Example

1. $\mathrm{P} \Rightarrow \sim \mathrm{Q} \vee \mathrm{R} \Leftrightarrow \mathrm{S}$ is an abbr. for $\big(\mathrm{P} \Rightarrow \big[(\sim \mathrm{Q}) \vee \mathrm{R}\big]\big) \Leftrightarrow \mathrm{S}$.
2. $\mathrm{P} \vee \sim \mathrm{Q} \Leftrightarrow \mathrm{R} \Rightarrow \mathrm{S}$ is an abbr. for $\big[\mathrm{P} \vee (\sim \mathrm{Q})\big] \Leftrightarrow (\mathrm{R} \Rightarrow \mathrm{S})$.
3. $\mathrm{P} \Rightarrow \mathrm{Q} \Rightarrow \mathrm{R}$ is an abbr. for $(\mathrm{P} \Rightarrow \mathrm{Q}) \Rightarrow \mathrm{R}$.

§1.3 Quantified Statements

Definition

An ***open sentence*** is a sentence that contains variables. When P is an open sentence with a variable x (or variables $x_1, \cdots, x_n$), the sentence is symbolized by $\mathrm{P}(x)$ (or $\mathrm{P}(x_1, \cdots, x_n)$).
The ***truth set*** of an open sentence is the collection of variables (from a certain universe) that may be substituted to make the open sentence a true proposition. (使得 $\mathrm{P}(x)$ 為真的所有 x 形成 the truth set of $\mathrm{P}(x)$)

Remark:

In general, **an open sentence is not a proposition**. It can be true or false depending on the value of variables.

§1.3 Quantified Statements

Example

Let $\mathrm{P}(x)$ be the open sentence "x is a prime number between 5060 and 5090". In this open sentence, the universe is usually chosen to be $\mathbb{N}$, the natural number system, and the truth set of $\mathrm{P}(x)$ is $\{5077, 5081, 5087\}$.

Remark:

The truth set of an open sentence $\mathrm{P}(x)$ depends on the universe where x belongs to. For example, suppose that $\mathrm{P}(x)$ is the open sentence "$x^2 + 1 = 0$". If the universe is $\mathbb{R}$, then $\mathrm{P}(x)$ is false for all x (in the universe). On the other hand, if the universe is $\mathbb{C}$, the complex plane, then $\mathrm{P}(x)$ is true when $x = \pm i$ (which also implies that the truth set of $\mathrm{P}(x)$ is $\{i, -i\}$).

§1.3 Quantified Statements

Definition

With a universe X specified, two open sentences $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are equivalent if they have the same truth set of all $x \in X$.

Example

The two sentences "$3x + 2 = 20$" and "$2x - 7 = 5$" are equivalent open sentences in any of the number system, such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

Example

The two sentences "$x^2 - 1 > 0$" and "$(x < -1) \vee (x > 1)$" are equivalent open sentences in $\mathbb{R}$.

§1.3 Quantified Statements

Given an open sentence $\mathrm{P}(x)$, the first question that we should ask ourself is "whether the truth set of $\mathrm{P}(x)$ is empty or not".

Definition

The symbol $\exists$ is called the ***existential quantifier***. For an open sentence $\mathrm{P}(x)$, the sentence $(\exists x)\mathrm{P}(x)$ is read "there exists x such that $\mathrm{P}(x)$" or "for some x, $\mathrm{P}(x)$". The sentence $(\exists x)\mathrm{P}(x)$ is true if the truth set of $\mathrm{P}(x)$ is non-empty.

Remark:

An open sentence $\mathrm{P}(x)$ does **not** have a truth value, but the quantified sentence $(\exists x)\mathrm{P}(x)$ does.

§1.3 Quantified Statements

Example

The quantified sentence $(\exists\, x)(x^7 - 12x^3 + 16x - 3 = 0)$ is true in the universe of real numbers.

Example (Fermat number)

The quantified sentence $(\exists\, n)(2^{2^n} + 1$ is a prime number$)$ is true in the universe of natural numbers.

Example (Fermat's last theorem)

The quantified sentence

$$(\exists\, x, y, z, n)(x^n + y^n = z^n \wedge n \geqslant 3)$$

is true in the universe of integers, but is false in the universe of natural numbers.

§1.3 Quantified Statements

Definition

The symbol $\forall$ is called the ***universal quantifier***. For an open sentence $\mathrm{P}(x)$, the sentence $(\forall\, x)\mathrm{P}(x)$ is read "for all x, $\mathrm{P}(x)$", "for every x, $\mathrm{P}(x)$" or "for every given x (in the universe), $\mathrm{P}(x)$". The sentence $(\forall\, x)\mathrm{P}(x)$ is true if the truth set of $\mathrm{P}(x)$ is the entire universe.

Example

The quantified sentence $(\forall\, n)(2^{2^n}+1$ is a prime number) is false in the universe of natural numbers since

$$2^{2^6}+1=641\times 6700417\,.$$

§1.3 Quantified Statements

In general, statements of the form "every element of the set A has the property P" and "some element of the set A has property P" may be symbolized as $(\forall\, x \in A)\mathrm{P}(x)$ and $(\exists\, x \in A)\mathrm{P}(x)$, respective. Moreover,

1. "All $\mathrm{P}(x)$ are $\mathrm{Q}(x)$" (所有滿足 P 的 x 都滿足 Q or 只要滿足 P 的 x 就滿足 Q) should be symbolized as

$$\text{“}(\forall\, x)\big(\mathrm{P}(x) \Rightarrow \mathrm{Q}(x)\big)\text{”}.$$

(See the next slide for the explanation!)

2. "Some $\mathrm{P}(x)$ are $\mathrm{Q}(x)$" (有些滿足 P 的 x 也滿足 Q or 有些 x 同時滿足 P 和 Q) should be symbolized as

$$\text{“}(\exists\, x)\big(\mathrm{P}(x) \wedge \mathrm{Q}(x)\big)\text{”}.$$