基礎數學 MA-1015A

Chapter 1. Logic and Proofs

§1.1 Propositions and Connectives

Definition

A ***proposition*** is a sentence that has exactly one truth value. It is either true, which we denote by T, or false, which we denote by F.

Example

$7^2 > 60$ (F), $\pi > 3$ (T), Earth is the closest planet to the sun (F).

Example

The statement "the north Pacific right whale（露脊鯨）will be extinct species before the year 2525" has one truth value but it takes time to determine the truth value.

Example

That "Euclid was left-handed" is a statement that has one truth value but may never be known.

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Definition

A ***negation*** of a proposition P, denoted by $\sim\mathrm{P}$, is the proposition "not P". The proposition $\sim\mathrm{P}$ is true / false exactly when P is false / true.

Definition

Given propositions P and Q, the ***conjunction*** / ***disjunction*** of P and Q, denoted by $\mathrm{P} \wedge \mathrm{Q}$ / $\mathrm{P} \vee \mathrm{Q}$, is the proposition "$\mathrm{P}$ and / or Q". $\mathrm{P} \wedge \mathrm{Q}$ / $\mathrm{P} \vee \mathrm{Q}$ is true exactly when both P and Q are true / at least one of P or Q is true.

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Example

Now we analyze the sentence "either 7 is prime and 9 is even, or else 11 is not less than 3". Let P denote the sentence "7 is a prime", Q denote the sentence "9 is even", and R denote the sentence "11 is less than 3". Then the original sentence can be symbolized by $(\mathrm{P} \wedge \mathrm{Q}) \vee (\sim \mathrm{R})$, and the table of truth value for this sentence is

P	Q	R	$\mathrm{P} \wedge \mathrm{Q}$	$\sim \mathrm{R}$	$(\mathrm{P} \wedge \mathrm{Q}) \vee (\sim \mathrm{R})$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
T	F	F	F	T	T
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Since P is true and Q, R are false, the sentence $(\mathrm{P} \wedge \mathrm{Q}) \vee (\sim \mathrm{R})$ is true.

§1.1 Propositions and Connectives

Definition

A ***tautology*** / ***contradiction*** is a propositional form that is true / false for every assignment of truth values to its component.

Example

The logic symbol $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

Example

The logic symbol $\sim(P \vee \sim P) \vee (Q \wedge \sim Q)$ is a contradiction.

Definition

Two propositional forms are said to be ***equivalent*** if they have the same truth value.

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Theorem

For propositions P, Q, R, *we have the following:*

(a) $\mathrm{P} \Leftrightarrow \sim(\sim \mathrm{P})$. (**Double Negation Law**)

(b) $\mathrm{P} \vee \mathrm{Q} \Leftrightarrow \mathrm{Q} \vee \mathrm{P}$

(c) $\mathrm{P} \wedge \mathrm{Q} \Leftrightarrow \mathrm{Q} \wedge \mathrm{P}$ (**Commutative Laws**)

(d) $\mathrm{P} \vee (\mathrm{Q} \vee \mathrm{R}) \Leftrightarrow (\mathrm{P} \vee \mathrm{Q}) \vee \mathrm{R}$

(e) $\mathrm{P} \wedge (\mathrm{Q} \wedge \mathrm{R}) \Leftrightarrow (\mathrm{P} \wedge \mathrm{Q}) \wedge \mathrm{R}$ (**Associative Laws**)

(f) $\mathrm{P} \wedge (\mathrm{Q} \vee \mathrm{R}) \Leftrightarrow (\mathrm{P} \wedge \mathrm{Q}) \vee (\mathrm{P} \wedge \mathrm{R})$

(g) $\mathrm{P} \vee (\mathrm{Q} \wedge \mathrm{R}) \Leftrightarrow (\mathrm{P} \vee \mathrm{Q}) \wedge (\mathrm{P} \vee \mathrm{R})$ (**Distributive Laws**)

(h) $\sim(\mathrm{P} \wedge \mathrm{Q}) \Leftrightarrow (\sim \mathrm{P}) \vee (\sim \mathrm{Q})$

(i) $\sim(\mathrm{P} \vee \mathrm{Q}) \Leftrightarrow (\sim \mathrm{P}) \wedge (\sim \mathrm{Q})$ (**De Morgan's Laws**)

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Proof.

We prove (g) for example, and the other cases can be shown in a similar fashion. Using the truth table,

P	Q	R	$Q\wedge R$	$P\vee(Q\wedge R)$	$P\vee Q$	$P\vee R$	$(P\vee Q)\wedge(P\vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

we find that "$P \vee (Q \wedge R)$" is equivalent to "$(P \vee Q) \wedge (P \vee R)$". □

§1.1 Propositions and Connectives

Definition

A ***denial*** of a proposition is any proposition equivalent to $\sim \mathrm{P}$.

- **Rules for $\sim$, $\wedge$ and $\vee$:**

1. $\sim$ is always applied to the smallest proposition following it.
2. $\wedge$ connects the smallest propositions surrounding it.
3. $\vee$ connects the smallest propositions surrounding it.

Example

Under the convention above, we have

1. $\sim \mathrm{P} \vee \sim \mathrm{Q} \Leftrightarrow (\sim \mathrm{P}) \vee (\sim \mathrm{Q})$.
2. $\mathrm{P} \vee \mathrm{Q} \vee \mathrm{R} \Leftrightarrow (\mathrm{P} \vee \mathrm{Q}) \vee \mathrm{R} \Leftrightarrow \mathrm{P} \vee (\mathrm{Q} \vee \mathrm{R})$.
3. $\mathrm{P} \wedge \sim \mathrm{Q} \vee \sim \mathrm{R} \Leftrightarrow \big[\mathrm{P} \wedge (\sim \mathrm{Q})\big] \vee (\sim \mathrm{R})$.
4. $\mathrm{R} \wedge \mathrm{P} \wedge \mathrm{S} \wedge \mathrm{Q} \Leftrightarrow \big[(\mathrm{R} \wedge \mathrm{P}) \wedge \mathrm{S}\big] \wedge \mathrm{Q}$.