基礎數學補充教材 May 13 2016

Proposition 0.1. *Let S be a non-empty set. The following three statements are equivalent:*

(a) *S is countable;*

(b) *there exists a surjection $f : \mathbb{N} \to S$;*

(c) *there exists an injection $f : S \to \mathbb{N}$.*

Proof. "(a) $\Rightarrow$ (b)" First suppose that $S = \{x_1, \cdots, x_n\}$ is finite. Define $f : \mathbb{N} \to S$ by

$$f(k) = \begin{cases} x_k & \text{if } k < n\,, \\ x_n & \text{if } k \geqslant n\,. \end{cases}$$

Then $f : \mathbb{N} \to S$ is a surjection. Now suppose that S is denumerable. Then by definition of countability, there exists $f : \mathbb{N} \xrightarrow[onto]{1-1} S$.

"(a) $\Leftarrow$ (b)" W.L.O.G. we assume that S is an infinite set. Let $k_1 = 1$. Since $\#(S) = \infty$, $S_1 \equiv S \backslash \{f(k_1)\} \neq \varnothing$; thus $N_1 \equiv f^{-1}(S_1)$ is a non-empty subset of $\mathbb{N}$. By the well-ordered principle (**WOP**) of $\mathbb{N}$, N_1 has a smallest element denoted by k_2. Since $\#(S) = \infty$, $S_2 = S \backslash \{f(k_1), f(k_2)\} \neq \varnothing$; thus $N_2 \equiv f^{-1}(S_2)$ is a non-empty subset of $\mathbb{N}$ and possesses a smallest element denoted by k_3. We continue this process and obtain a set $\{k_1, k_2, \cdots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \cdots$, and k_j is the smallest element of $N_{j-1} \equiv f^{-1}(S \backslash \{f(k_1), f(k_2), \cdots, f(k_{j-1})\})$.

Claim: $f : \{k_1, k_2, \cdots\} \to S$ is one-to-one and onto.

Proof of claim: The injectivity of f is easy to see since $f(k_j) \notin \big\{f(k_1), f(k_2), \cdots, f(k_{j-1})\big\}$ for all $j \geqslant 2$. For surjectivity, assume that there is $s \in S$ such that $s \notin f(\{k_1, k_2, \cdots\})$. Since $f : \mathbb{N} \to \mathbb{S}$ is onto, $f^{-1}(\{s\})$ is a non-empty subset of $\mathbb{N}$; thus possesses a smallest element k. Since $s \notin f(\{k_1, k_2, \cdots\})$, there exists $\ell \in \mathbb{N}$ such that $k_\ell < k < k_{\ell+1}$. As a consequence, there exists $k \in N_\ell$ such that $k < k_{\ell+1}$ which contradicts to the fact that $k_{\ell+1}$ is the smallest element of N_ℓ.

Define $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ by $g(j) = k_j$. Then $g : \mathbb{N} \to \{k_1, k_2, \cdots\}$ is one-to-one and onto; thus $h = g \circ f : \mathbb{N} \xrightarrow[onto]{1-1} S$.

"(a) $\Rightarrow$ (c)" If $S = \{x_1, \cdots, x_n\}$ is finite, we simply let $f : S \to \mathbb{N}$ be $f(x_n) = n$. Then f is clearly an injection. If S is denumerable, by definition there exists $g : \mathbb{N} \xrightarrow[onto]{1-1} S$ which suggests that $f = g^{-1} : S \to \mathbb{N}$ is an injection.

"(a) $\Leftarrow$ (c)" Let $f : S \to \mathbb{N}$ be an injection. If f is also surjective, then $f : S \xrightarrow[onto]{1-1} \mathbb{N}$ which implies that S is denumerable. Now suppose that $f(S) \subsetneq \mathbb{N}$. Since S is non-empty, there exists $s \in S$. Let $g : \mathbb{N} \to S$ be defined by

$$g(n) = \begin{cases} f^{-1}(n) & \text{if } n \in f(S)\,, \\ s & \text{if } n \notin f(S)\,. \end{cases}$$

Then clearly $g : \mathbb{N} \to S$ is surjective; thus the equivalence between (a) and (b) implies that S is countable. $\square$