Exercise Problem Sets 14

May. 19. 2023

In the following problems, H denotes the Heaviside function defined by

$$H(x) = \mathbf{1}_{[0,\infty)}(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ 1 & \text{if } x \geqslant 0\,. \end{cases}$$

Problem 1. 1. Let $a > 0$ be a constant. Find the Fourier transform of the function $f(x) = e^{-ax}H(x)$.

2. Find the Fourier transform of the functions $g(x) = \dfrac{1}{2 - 3ix - x^2}$ by

   (a) Rewriting g as the sum of two fractions and apply the result in part 1.

   (b) Convolution.

Problem 2. Find the Fourier transform of the function $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| \leqslant 1\,, \\ 0 & \text{if } |x| > 1\,, \end{cases}$ and hence evaluate

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx\,.$$

Problem 3. Find the Fourier transform of the function $f(x) = \begin{cases} a - |x| & \text{if } |x| \leqslant a\,, \\ 0 & \text{if } |x| > a\,, \end{cases}$ and hence prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}\,.$$

Problem 4. Solve the integral equation $\displaystyle\int_0^\infty f(x)\cos(\lambda x)\, dx = \begin{cases} 1 - \lambda & \text{if } 0 \leqslant \lambda \leqslant 1\,, \\ 0 & \text{if } \lambda > 1\,. \end{cases}$ Hence deduce that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$.

Problem 5. 1. Let $\alpha > 0$. Compute the Fourier transform of the function

$$f_\alpha(x) = \begin{cases} e^{-\alpha x} & \text{if } x \geqslant 0\,, \\ -e^{\alpha x} & \text{if } x < 0\,. \end{cases}$$

2. Show that $\lim\limits_{\alpha\to 0^+} \widehat{f}_\alpha(\xi) = \widehat{\text{sgn}}(\xi)$; that is,

$$\lim_{\alpha\to 0^+} \langle \widehat{f}_\alpha, \phi\rangle = \langle \widehat{\text{sgn}}(\xi), \phi\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

   where sgn is the sign function given by

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0\,, \\ -1 & \text{if } x < 0\,, \\ 0 & \text{if } x = 0\,. \end{cases} \tag{0.1}$$

3. Find the Fourier transform of the Heaviside function H using $H = \dfrac{1}{2}(1 + \text{sgn})$.

4. Use the Fourier transform of sgn to compute the Fourier transform of the following functions:

   (a) $f_1(t) = \dfrac{1}{t}$.   (b) $f_2(t) = |t|$.

   Use (a) and (b) to compute the Fourier transform of

   (c) $f_3(t) = -\dfrac{1}{t^2}$.   (d) $f_4(t) = \dfrac{2}{t^2}$.

Problem 6. In this problem we discuss the derivative of tempered distributions. Complete the following.

1. Show that
$$\Big\langle \frac{\partial f}{\partial x_j}, g\Big\rangle = -\Big\langle f, \frac{\partial g}{\partial x_j}\Big\rangle \qquad \forall\, f, g \in \mathscr{S}(\mathbb{R}^n)\,.$$
This leads to the definition of the derivatives of tempered distributions: Let $T \in \mathscr{S}(\mathbb{R}^n)'$ be a tempered distribution. The partial derivative of T w.r.t. x_j, denoted by $\dfrac{\partial T}{\partial x_j}$, is a tempered distribution defined by
$$\Big\langle \frac{\partial T}{\partial x_j}, \phi\Big\rangle = -\Big\langle T, \frac{\partial \phi}{\partial x_j}\Big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$
Verify that $\dfrac{\partial T}{\partial x_j}$ is indeed a tempered distribution; that is, show that there exists a sequence $\{C_k\}_{k=1}^{\infty}$ such that
$$\Big|\Big\langle \frac{\partial T}{\partial x_j}, \phi\Big\rangle\Big| \leqslant C_k p_k(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n) \text{ and } k \gg 1\,.$$

2. Show that for $1 \leqslant j \leqslant n$,
$$\mathscr{F}_x\Big[\frac{\partial T}{\partial x_j}\Big](\xi) = i\xi_j \widehat{T}(\xi) \qquad \text{and} \qquad \frac{\partial}{\partial x_j}\widehat{T}(\xi) = -i\mathscr{F}_x[xT(x)](\xi)$$
or to be more precise,
$$\Big\langle \widehat{\frac{\partial T}{\partial x_j}}, \phi\Big\rangle = \big\langle \widehat{T}(\xi), i\xi_j \phi(\xi)\big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)$$
and
$$\Big\langle \frac{\partial}{\partial \xi_j}\widehat{T}(\xi), \phi(\xi)\Big\rangle = \big\langle T(x), -ix\widehat{\phi}(x)\big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$
In other words, the Fourier transform of derivatives of tempered distributions still obeys Lemma 9.9 and 9.11 in the lecture note.

Problem 7. Let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ be the sign function given by (0.1). Then clearly sgn is a tempered distribution since
$$\big|\langle \operatorname{sgn}, \phi\rangle\big| \leqslant \|\phi\|_{L^1(\mathbb{R})} \leqslant \pi p_2(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$
Show that $\dfrac{d}{dx}\operatorname{sgn}(x) = 2\delta$ in $\mathscr{S}(\mathbb{R})'$, where the derivative of tempered distributions is defined in Problem 6 and δ is the Dirac delta function.

Problem 8. Compute the Fourier transform of the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = |x|^\alpha$, where $-n < \alpha < 0$, by the following procedure.

1. Show that $f \notin L^1(\mathbb{R}^n)$.

2. Recall that the Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$. Show that

$$|x|^\alpha = \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\, ds \qquad \forall\, x \neq 0\,.$$

3. Find that Fourier transform of f.

4. Find the Fourier transform of the function $g : \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = x_1|x|^\alpha$, where x_1 is the first component of x and $-n-2 < \alpha < -2$.

Hint: 3. Compute $\langle |x|^\alpha, \widehat{\phi}(x)\rangle$ by applying Fubini's Theorem several times.

4. Note that $g(x) = \dfrac{1}{\alpha+2}\dfrac{\partial}{\partial x_1}|x|^{\alpha+2}$ so that you can apply the results above. See Problem 6 for the Fourier transform of derivatives of tempered distributions.

Problem 9. Let $f \in L^1(\mathbb{R})$. Show that the function $y = \int_{-\infty}^x f(t)\, dt$ can be written as the convolution of f and a function $\phi \in L^1_{\text{loc}}(\mathbb{R})$.

Problem 10. In this problem we use symbolic computation to find the Fourier transform of the function

$$f(x) = \begin{cases} \dfrac{\sin(\omega x)}{x} & \text{if } x \neq 0\,, \\ \omega & \text{if } x = 0\,, \end{cases}$$

without knowing that it is the Fourier transform of the function $y = \sqrt{\frac{\pi}{2}}\mathbf{1}_{[-\omega,\omega]}(x)$. Complete the following.

1. Note that $f \notin L^1(\mathbb{R})$ but $f \in \mathscr{S}(\mathbb{R}^n)'$. Therefore, $\widehat{f} \in \mathscr{S}(\mathbb{R})$. Find the derivative of $\widehat{f}$, where the derivatives of tempered distributions is given in Problem 6.

2. Suppose that you can use the Fundamental Theorem of Calculus so that

$$\widehat{f}(\xi) - \widehat{f}(0) = \int_0^\xi \widehat{f}\,'(t)\, dt\,.$$

   Note that in previous exercise you are asked to show that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. Use this fact and treat $\delta_{\pm\omega}$ as the evaluation operation at $\pm\omega$ to find $\widehat{f}(\xi)$ (for $\xi \neq \pm\omega$).

Hint: 1. Recall that we have shown that $\mathscr{F}_x[\sin(\omega x)](\xi) = \dfrac{\sqrt{2\pi}}{2i}(\delta_\omega - \delta_{-\omega})$.

Problem 11. Let ω be a positive real number, and $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{\sin(\omega|x|)}{|x|} & \text{if } x \neq 0\,, \\ \omega & \text{if } x = 0\,, \end{cases}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ if $x = (x_1, x_2, x_3)$. In this problem we are concerned with the Fourier transform of f. Complete the following.

1. Show that $f \in \mathscr{S}(\mathbb{R}^3)'$.

2. Show that the Fourier transform of f is given by

$$\langle \widehat{f}, \varphi \rangle = \sqrt{\frac{\pi}{2}} \frac{1}{\omega} \int_{\partial B(0,\omega)} \varphi\, dS$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^3)$, where $\displaystyle\int_{\partial B(0,\omega)} \varphi\, dS$ is the surface integral of φ on the sphere $\partial B(0, \omega)$ defined by

$$\int_{\partial B(0,\omega)} \varphi\, dS \equiv \int_0^{\pi} \int_0^{2\pi} \varphi(\omega\cos\theta\sin\phi, \omega\sin\theta\sin\phi, \omega\cos\phi)\omega^2 \sin\phi\, d\theta d\phi\,.$$

Hint of 2: You can show part 2 through the following procedures:

Step 1: By the definition of the Fourier transform of the tempered distributions,

$$\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle = \lim_{m\to\infty} \int_{B(0,m)} f(x) \Big(\frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \varphi(\xi) e^{-ix\cdot\xi}\, d\xi \Big)\, dx$$

and the Fubini Theorem implies that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}^3} \lim_{m\to\infty} \int_{\mathbb{R}^3} \Big(\int_{B(0,m)} f(x) e^{-ix\cdot\xi}\, dx \Big) \varphi(\xi) d\xi\,.$$

We focus on the inner integral first. Show that for each 3×3 orthonormal matrix O,

$$\int_{B(0,m)} f(x) e^{-ix\cdot\xi}\, dx = \int_{B(0,m)} \frac{\sin(\omega|y|)}{|y|} e^{-i(\mathrm{O}^{\mathrm{T}}\xi)\cdot y}\, dy\,.$$

Step 2: For each $\xi \in \mathbb{R}^3$, choose a 3×3 orthonormal matrix O such that $\mathrm{O}^{\mathrm{T}}\xi = (0, 0, |\xi|)$. Using the spherical coordinate $y = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$ to show that

$$\int_{B(0,m)} f(x) e^{-ix\cdot\xi}\, dx = \int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|}\, d\rho$$

so that we conclude that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}^3} \lim_{m\to\infty} \int_{\mathbb{R}^3} \Big(\int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|} \varphi(\xi)\, d\rho \Big) d\xi\,.$$

Step 3: For each $r > 0$, define $\psi(r)$ as the surface integral of φ on $\partial B(0,r)$; that is,

$$\psi(r) = \int_{\partial B(0,r)} \varphi\, dS \equiv \int_0^\pi \int_0^{2\pi} \varphi(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) r^2 \sin\phi\, d\theta d\phi\,.$$

Using the spherical coordinate $\xi = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ to show that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big(\int_0^\infty \sin(\omega\rho)\sin(r\rho) \frac{2\psi(r)}{r}\, dr \Big) d\rho\,.$$

Step 4: Apply the conclusion in Problem 3 of Exercise 12.

Problem 12. 1. Show that the function $R : \mathbb{R} \to \mathbb{R}$ given by

$$\mathrm{R}(x) = \begin{cases} x & \text{if } x \geqslant 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

is a tempered distribution.

2. Let T be a generalized function defined by

$$\langle T, \phi \rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\phi(x)}{x}\, dx = \lim_{\epsilon \to 0^+} \Big(\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty \Big) \frac{\phi(x)}{x}\, dx \qquad \forall\, \phi \in \mathscr{C}_c^\infty(\mathbb{R})\,.$$

Show that $T \in \mathscr{S}(\mathbb{R})'$.

3. Let H be the Heaviside function given by

$$H(x) = \begin{cases} 0 & \text{if } x \leqslant 0\,, \\ 1 & \text{if } x > 0\,. \end{cases}$$

Show that $\widehat{H} = \dfrac{-i}{\sqrt{2\pi}} T + \sqrt{\dfrac{\pi}{2}}\,\delta$, here δ is the Dirac delta function.

Hint: 3. Let $G(x) = \exp\big(-\frac{x^2}{2}\big)$. For each $\phi \in \mathscr{S}(\mathbb{R})$, define $\psi = \phi - \phi(0)G$ (which belongs to $\mathscr{S}(\mathbb{R})$). Use the identity

$$\langle \widehat{H}, \phi \rangle = \langle H, \widehat{\psi} \rangle - \phi(0) \langle H, \widehat{G} \rangle$$

to make the conclusion.

Problem 13. The Hilbert transform of a function $f : \mathbb{R} \to \mathbb{R}$, denoted by $\mathscr{H}[f]$, is a function defined (formally) by

$$\mathscr{H}[f](x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|y-x|>\epsilon} \frac{f(y)}{x-y}\, dy\,,$$

1. Show that $\mathscr{H}[f]$ is well-defined if $f \in \mathscr{S}(\mathbb{R})$.

2. Show that $\mathscr{F}\big[\mathscr{H}[f]\big](\xi) = i\mathrm{sgn}(\xi)\widehat{f}(\xi)$ for all $f \in \mathscr{S}(\mathbb{R})$.

3. Show that $\big\|\mathscr{H}[f]\big\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ for all $f \in \mathscr{S}(\mathbb{R})$, where $\|g\|_{L^2(\mathbb{R})} = \Big(\int_{\mathbb{R}} |g(x)|^2\, dx \Big)^{\frac{1}{2}}$.

Hint: Consider the tempered distribution T defined in Problem 12 by

$$\langle T, \phi \rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\phi(x)}{x}\, dx = \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{\phi(x)}{x}\, dx \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$

1. Show that $\mathscr{H}[f] = \langle T, \tau_x \widetilde{f} \rangle$ for all $f \in \mathscr{S}(\mathbb{R})$, where τ_x is a translation operator.

2. Show that the tempered distribution S defined by $\langle S, \phi \rangle = \langle T(x), x\phi(x) \rangle$ is indeed the same as the tempered distribution

$$\phi \mapsto \int_{\mathbb{R}} \phi(x)\, dx = \langle 1, \phi \rangle\,.$$

   Use Problem 6 to show that $\frac{d}{d\xi}\widehat{T}(\xi) = -\sqrt{\frac{\pi}{2}}\, i \frac{d}{d\xi}\mathrm{sgn}(\xi)$, where sgn is given in Problem 7. Use the fact that $\frac{dT}{dx} = 0$ if and only if there exists C such that $\langle T, \phi \rangle = \langle C, \phi \rangle$ for all $\phi \in \mathscr{S}(\mathbb{R})$ to conclude that

$$\widehat{T}(\xi) = -\sqrt{\frac{\pi}{2}}\, i\mathrm{sgn}(\xi) + C$$

   for some constant C. Find the constant C and also show that $\mathscr{H}[f] = \frac{1}{\pi} T * f = \sqrt{\frac{2}{\pi}}\, T \star f$.

3. Use the Plancherel formula.