Exercise Problem Sets 13

May. 12. 2023

Problem 1. Suppose that the Fourier transform and inverse Fourier transform of a (Schwartz) function f are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i2\pi x\cdot\xi} dx \qquad \text{and} \qquad \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{i2\pi x\cdot\xi} d\xi\,. \tag{0.1}$$

In the previous exercise we have shown that $\check{\widehat{f}} = \widehat{\check{f}} = f$ if $f \in \mathscr{S}(\mathbb{R}^n)$. Complete the following.

1. Show the Poisson summation formula

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k) = \sum_{n=-\infty}^{\infty} f(n) \qquad \forall\, f \in \mathscr{S}(\mathbb{R})\,.$$

2. Suppose that $g \in \mathscr{S}(\mathbb{R})$. Show that

$$\sum_{k=-\infty}^{\infty} \widehat{g}(\xi - \frac{k}{T}) = T \sum_{n=-\infty}^{\infty} g(nT) e^{-i2\pi nT\xi} \qquad \forall\, \xi \in \mathbb{R}\,.$$

   In particular, if $g \in \mathscr{S}(\mathbb{R})$ and $\mathrm{supp}(\widehat{f}) \subseteq [-\frac{1}{T}, \frac{1}{T}]$,

$$\widehat{g}(\xi) = T \sum_{n=-\infty}^{\infty} g(nT) e^{-i2\pi nT\xi} \qquad \forall\, \xi \in [-\frac{1}{T}, \frac{1}{T}]\,.$$

   This implies that if $\widehat{g}$ has compact support in $[-\frac{1}{T}, \frac{1}{T}]$, g can be reconstructed based on partial knowledge of g, namely $\{g(nT)\}_{n=-\infty}^{\infty}$.

Hint of 1: Define $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(x) = \sum_{n=-\infty}^{\infty} f(n+x)$. Show that $\Phi \in \mathscr{C}^1(\mathbb{R};\mathbb{R})$ (using Theorem 7.32 in the lecture note) and Φ is periodic with period 1. Express Φ in terms of Fourier series and find that value $\Phi(0)$.

Hint of 2: For each $\xi \in \mathbb{R}$, let $f(x) = g(Tx)e^{-i2\pi Tx\xi}$ and use 1.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous integrable function. Show that if $\mathrm{supp}(\widehat{f}) \subseteq [-B, B]$, where $\widehat{f}$ is given by (0.1), then

$$f(x) = \sum_{k=-\infty}^{\infty} f\big(\frac{k}{2B}\big) \mathrm{sinc}(2Bx - k) \qquad \forall\, x \in \mathbb{R}\,, \tag{0.2}$$

where sinc is the normalized sinc function given by

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,, \end{cases}$$

Hint: Treat $\widehat{f}$ as a continuous function defined on $[-B, B]$. For each $x \in \mathbb{R}$, the Fourier inversion formula implies that

$$f(x) = \int_{-B}^{B} \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

so that the Fourier coefficients of $\widehat{f}$ is $\left\{\frac{1}{2B} f(\frac{-k}{2B})\right\}_{k=-\infty}^{\infty}$. Apply the inner product version of Parseval's identity (that is, (8.5.5) in the lecture note) to functions $\widehat{f}$ and g given by $g(\xi) = e^{-2\pi i x\xi}$ to express the integral above in terms of an infinite series.

Remark 0.1. The result of this problem is called the ***Shannon Interpolation Theorem***, and (0.2) is called the ***Whittaker–Shannon interpolation formula***.

In the following problems, assume that you can use the Fourier inversion formula (without checking that whether f and $\widehat{f}$ belong to $L^1(\mathbb{R}^n)$).

Problem 3. Use the Fourier transform to find a function f such that

$$\int_{-\infty}^{\infty} f(x-y) e^{-|y|} dy = 2e^{-|x|} - e^{-2|x|} .$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \int_0^2 \frac{\sqrt{t}}{1+t} e^{ixt}\, dt .$$

Compute the following.

(a) The Fourier transform of f. (b) $\displaystyle\int_{-\infty}^{\infty} f(x)\cos x\, dx$. (c) $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\, dx$.

Problem 5. For a and b positive, use the Fourier transform to compute the integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(b(u-t))}{t(u-t)}\, dt$. (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(at)\sin(bt)}{t^2}\, dt$.

Problem 6. Use the Fourier transform to compute $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\, dx$.

Problem 7. Suppose that $f : \mathbb{R} \to \mathbb{C}$ has the Fourier transform $\widehat{f}(\xi) = \dfrac{1-i\xi}{1+i\xi}\dfrac{\sin\xi}{\xi}$. Compute

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx .$$

Problem 8. Suppose that $f : \mathbb{R} \to \mathbb{C}$ has the Fourier transform $\widehat{f}(\xi) = \dfrac{1}{|\xi|^3+1}$. Compute

$$\int_{-\infty}^{\infty} |(f * f')(x)|^2\, dx .$$

Problem 9. Let function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \displaystyle\int_0^1 \sqrt{t} e^{t^2} \cos(xt)\, dt$. Compute

$$\int_{-\infty}^{\infty} \left|f'(x)\right|^2\, dx .$$

Problem 10. Let $a > 0$. Use the Fourier transforms of sinc and sinc^2, together with the basic tools of Fourier transform theory, such as Plancherel's identity, substitution, and etc. to show the following. (Use only rules from Fourier transform theory. You should not do any detailed computation such as integration by parts.)

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^3(ax)}{x^3}\,dx = \frac{3a^2\pi}{4}$. (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^4(ax)}{x^4}\,dx = \frac{2a^3\pi}{3}$.

Problem 11. Deduce what you can about the Fourier transform $\widehat{f}(\xi)$ if you know that f satisfies the dilation equation

$$f(t) = f(2t) + f(2t-1) \qquad \forall\, t \in \mathbb{R}\,.$$