Exercise Problem Sets 10

Apr. 21. 2023

Problem 1. Use the Fourier series of the function $f : (-\pi, \pi) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & -\pi < x < 0\,, \\ \pi - x & 0 \leqslant x < \pi\,, \end{cases}$$

and compute

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^4}\,.$$

Solution. From Problem 2 of Exercise 9, we find that

$$s_k = \frac{1}{k} \quad \forall\, k \in \mathbb{N}\,, \qquad c_k = \frac{1-(-1)^k}{k^2\pi} \quad \forall\, k \in \mathbb{N} \qquad \text{and} \qquad c_0 = \frac{\pi}{2}\,.$$

Since

$$\int_{-\pi}^{\pi} \big|f(x)\big|^2\, dx = \int_0^{\pi} (\pi - x)^2\, dx = -\frac{1}{3}(\pi - x)^3\Big|_{x=0}^{x=\pi} = \frac{\pi^3}{3}$$

and

$$\sum_{k=1}^{\infty} s_k^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}\,, \qquad \sum_{k=1}^{\infty} c_k^2 = \sum_{k=1}^{\infty} \frac{2^2}{(2k-1)^4\pi^2} = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\,,$$

the Parseval identity implies that

$$\frac{\pi^2}{6} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x)\big|^2\, dx = \frac{c_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\big(c_k^2 + s_k^2\big) = \frac{\pi^2}{16} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} + \frac{\pi^2}{12}\,.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^2}{2}\Big(\frac{\pi^2}{6} - \frac{\pi^2}{12} - \frac{\pi^2}{16}\Big) = \frac{\pi^4}{96}\,.$$

Note that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \sum_{k=1}^{\infty}\frac{1}{(2k)^4} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^4} + \frac{\pi^4}{96}\,;$$

thus rearranging terms we conclude that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}\,.$$

□

Problem 2. Use the Fourier series of the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = x^3 - \pi^2 x$ to find the values of $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^3}$ and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^6}$.

Solution. Let $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficients of f. Note that f is an odd function; thus $c_k = 0$ for all $k \in \mathbb{N}$. On the other hand, for $k \in \mathbb{N}$ we have

$$\int_0^{\pi} x \sin kx\, dx = \frac{-x\cos kx}{k}\Big|_{x=0}^{x=\pi} + \frac{1}{k}\int_0^{\pi}\cos kx\, dx = \frac{\pi(-1)^{k+1}}{k}$$

and

$$
\begin{aligned}
\int_0^\pi x^3 \sin kx\, dx &= \frac{-x^3\cos kx}{k}\Big|_{x=0}^{x=\pi} + \frac{3}{k}\int_0^\pi x^2\cos kx\, dx \\
&= \frac{\pi^3(-1)^{k+1}}{k} + \frac{3}{k}\Big[\frac{x^2\sin kx}{k}\Big|_{x=0}^{x=\pi} - \frac{2}{k}\int_0^\pi x\sin kx\, dx\Big] \\
&= \frac{\pi^3(-1)^{k+1}}{k} - \frac{6}{k^2}\int_0^\pi x\sin kx\, dx \\
&= \frac{\pi^3(-1)^{k+1}}{k} - \frac{6}{k^2}\cdot\frac{\pi(-1)^{k+1}}{k}\,.
\end{aligned}
$$

Therefore, the computations above show that

$$
s_k = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^3-\pi^2 x)\sin kx\, dx = \frac{2}{\pi}\int_0^\pi (x^3-\pi^2 x)\sin kx\, dx = \frac{2}{\pi}\cdot\frac{6}{k^2}\cdot\frac{\pi(-1)^k}{k^3} = \frac{12(-1)^k}{k^3}
$$

so that the Fourier series of f is given by

$$
s(f,x) = \sum_{k=1}^{\infty}\frac{12(-1)^k}{k^3}\sin kx\,.
$$

Therefore, by the fact that h is Hölder continuous, by Theorem 8.17 in the lecture note we have

$$
\begin{aligned}
\frac{\pi^3}{8} - \pi^2\cdot\frac{\pi}{2} = h\big(\frac{\pi}{2}\big) = s\big(h,\frac{\pi}{2}\big) &= \sum_{k=1}^{\infty}\frac{12(-1)^k}{k^3}\sin\frac{k\pi}{2} = \sum_{k=1}^{\infty}\frac{-12}{(2k-1)^3}\sin\frac{(2k-1)\pi}{2} \\
&= 12\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^3}\,;
\end{aligned}
$$

thus

$$
\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^3} = -\frac{\pi^3}{32}\,.
$$

Moreover, the Parseval identity implies that

$$
\frac{1}{2}\sum_{k=1}^{\infty}|s_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^3-\pi^2 x)^2\, dx\,;
$$

thus

$$
\sum_{k=1}^{\infty}\frac{1}{k^6} = \frac{1}{144\pi}\int_{-\pi}^{\pi}(x^3-\pi^2 x)^2\, dx = \frac{1}{72\pi}\int_0^\pi (x^6 - 2\pi^2x^4 + \pi^4x^2)\, dx = \frac{\pi^6}{945}\,.
$$

□

Problem 3. For each $n \in \mathbb{Z}$, define the Bessel functions $J_n(x)$ through the Fourier series by

$$
e^{ix\sin t} = \sum_{n=-\infty}^{\infty} J_n(x)e^{int}\,.
$$

Compute $\sum\limits_{n=-\infty}^{\infty}|J_n(x)|^2$ for $x \in \mathbb{R}$.

Proof. For a fixed $x \in \mathbb{R}$, by treating the function $y = e^{ix\sin t}$ as a 2π-periodic function of t, we find that the Fourier series of the function is given by

$$\sum_{n=-\infty}^{\infty} J_n(x)e^{int},$$

where $\{J_n(x)\}_{n=-\infty}^{\infty}$ is the Fourier coefficients given by

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin t}e^{-int}\,dt.$$

By the Parseval identity,

$$\sum_{n=-\infty}^{\infty} |J_n(x)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|e^{ix\sin t}\right|^2 dt = 1. \qquad \square$$

Problem 4. Let $f : [0, L] \to \mathbb{R}$ be a square integrable function.

1. Suppose that $\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\frac{k\pi x}{L}$ is the cosine series of f. Find $\sum_{k=1}^{\infty} c_k^2$ in terms of integrals of f and f^2.

2. Suppose that $\sum_{k=1}^{\infty} s_k \sin\frac{k\pi x}{L}$ is the sine series of f. Find $\sum_{k=1}^{\infty} s_k^2$ in terms of integral of f^2.

Proof. 1. Let $f_e : [-L, L] \to \mathbb{R}$ be the even extension of f. Then

$$s(f_e, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\frac{k\pi x}{L},$$

where

$$c_k = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{k\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos kx\,dx.$$

In particular, $c_0 = \frac{1}{L}\int_{-L}^{L} f_e(x)\,dx = \frac{2}{L}\int_0^L f(x)\,dx$. By the Parseval identity,

$$\frac{c_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty} c_k^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_e\Big(\frac{Lx}{\pi}\Big)^2 dx = \frac{1}{2L}\int_{-L}^{L} f_e(x)^2\,dx = \frac{1}{L}\int_0^L f(x)^2\,dx.$$

Therefore,

$$\sum_{k=1}^{\infty} c_k^2 = 2\Big(\frac{1}{L}\int_0^L f(x)^2\,dx - \frac{c_0^2}{4}\Big) = \frac{2}{L}\int_0^L f(x)^2\,dx - \frac{1}{2L^2}\Big(\int_0^L f(x)\,dx\Big)^2.$$

2. Let $f_o : [-L, L] \to \mathbb{R}$ be the odd extension of f. Then

$$s(f_0, x) = \sum_{k=1}^{\infty} s_k \sin\frac{k\pi x}{L},$$

where

$$s_k = \frac{1}{L}\int_{-L}^{L} f_o(x)\sin\frac{k\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx.$$

By the Parseval identity,

$$\frac{1}{2}\sum_{k=1}^{\infty} s_k^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_o\Big(\frac{Lx}{\pi}\Big)^2\,dx = \frac{1}{2L}\int_{-L}^{L} f_o(x)^2\,dx = \frac{1}{L}\int_0^L f(x)^2\,dx\,.$$

Therefore,

$$\sum_{k=1}^{\infty} s_k^2 = \frac{2}{L}\int_0^L f(x)^2\,dx\,. \qquad \square$$

Problem 5. Expand the function $\cos x$ as a sine series on the interval $(0,\pi)$. Use the result to compute

$$\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2}\,.$$

How about expanding $\cos x$ as a sine series on the interval $(0,\pi/2)$?

Problem 6. This problem contributes to another proof of showing that the Fourier series of f converges uniformly to f on $\mathbb{R}$ if $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$ for $\frac{1}{2}<\alpha\leqslant 1$. Complete the following.

1. Let $f:\mathbb{R}\to\mathbb{R}$ be 2π-periodic such that f is Riemann integrable on $[-\pi,\pi]$. Show that

$$\widehat{f}_k = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(x+\frac{\pi}{k}\big)e^{-ikx}\,dx$$

and hence

$$\widehat{f}_k = \frac{1}{4\pi}\int_{-\pi}^{\pi}\big[f(x)-f\big(x+\frac{\pi}{k}\big)\big]e^{-ikx}\,dx\,.$$

Therefore, if $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies $|\widehat{f}_k|\leqslant\dfrac{\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}$.

2. Let $f:\mathbb{R}\to\mathbb{R}$ be 2π-periodic such that f is Riemann integrable on $[-\pi,\pi]$. Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x+h)-f(x-h)\big|^2\,dx = \sum_{k=-\infty}^{\infty}4\sin^2(kh)|\widehat{f}_k|^2\,.$$

Therefore, if $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies

$$\sum_{k=-\infty}^{\infty}\sin^2(kh)|\widehat{f}_k|^2\leqslant\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}2^{2(\alpha-1)}|h|^{2\alpha} \tag{0.1}$$

3. Let $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$, and $p\in\mathbb{N}$. Show that

$$\sum_{2^{p-1}\leqslant|k|<2^p}|\widehat{f}_k|^2\leqslant\frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^{2\alpha}}{2^{2\alpha p+1}}\,.$$

Hint: Let $h=\dfrac{\pi}{2^{p+1}}$ in (0.1).

4. Show that if $f\in\mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\frac{1}{2}<\alpha\leqslant 1$, then $\sum\limits_{k=-\infty}^{\infty}|\widehat{f}_k|<\infty$; thus Problem 1 of Exercise 9 implies that the Fourier series of f converges uniformly to f on $\mathbb{R}$.

Proof. 1. By substitution of variables,

$$\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy \overset{\text{``}y=x+\frac{\pi}{k}\text{''}}{=} \frac{1}{2\pi}\int_{-\pi-\frac{\pi}{k}}^{\pi-\frac{\pi}{k}} f\big(x+\frac{\pi}{k}\big)e^{-ikx}e^{-i\pi}\,dx$$

so that the periodicity of f and the function $y = e^{-ikx}$ implies that

$$\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(x+\frac{\pi}{k}\big)e^{-ikx}e^{-i\pi}\,dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(x+\frac{\pi}{k}\big)e^{-ikx}\,dx\,.$$

Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0,1]$. Then

$$\big|f(x)-f(y)\big| \leqslant \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}|x-y|^\alpha \qquad \forall\, x,y \in \mathbb{R}\,.$$

Therefore,

$$\big|f(x+\frac{\pi}{k}-f(x)\big| \leqslant \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\frac{\pi^\alpha}{k^\alpha}$$

and we then conclude that

$$\big|\widehat{f}_k\big| \leqslant \frac{1}{4\pi}\int_{-\pi}^{\pi}\big|f(x)-f\big(x+\frac{\pi}{k}\big)\big|\,dx \leqslant \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^\alpha}{4\pi k^\alpha}\int_{-\pi}^{\pi} dx = \frac{\pi^\alpha\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}\,.$$

2. For $h \neq 0$, let $g(x) = f(x+h) - f(x-h)$. Then by substitution of variables,

$$\begin{aligned}\widehat{g}_k &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(y+h)e^{-iky}\,dy - \int_{-\pi}^{\pi} f(y-h)e^{-iky}\,dy\Big]\\ &= \frac{1}{2\pi}\Big[\int_{-\pi+h}^{\pi+h} f(x)e^{-ikx}e^{ikh}\,dx - \int_{-\pi-h}^{\pi-h} f(x)e^{-ikx}e^{-ikh}\,dx\Big]\end{aligned}$$

so that the periodicity of f and the function $y = e^{-ikx}$ implies that

$$\begin{aligned}\widehat{g}_k &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(x)e^{-ikx}e^{ikh}\,dx - \int_{-\pi-h}^{\pi-h} f(x)e^{-ikx}e^{-ikh}\,dx\Big]\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\big(e^{ikh}-e^{-ikh}\big)\,dx = \frac{2i\sin(kh)}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = 2i\sin(kh)\widehat{f}_k\,.\end{aligned}$$

Therefore, the Parseval identity shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x+h)-f(x-h)\big|^2\,dx = \sum_{k=-\infty}^{\infty}\big|\widehat{g}_k\big|^2 = \sum_{k=-\infty}^{\infty} 4\sin^2(kh)\big|\widehat{f}_k\big|^2\,.$$

If in addition $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, then the identity above implies that

$$\sum_{k=-\infty}^{\infty} 4\sin^2(kh)\big|\widehat{f}_k\big|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}h^{2\alpha}\,dx = \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}(2h)^{2\alpha}$$

which verifies (0.1).

3. For each $p \in \mathbb{N}$, letting $h = \dfrac{\pi}{2^{p+1}}$ in (0.1) we find that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} \sin^2 \frac{k\pi}{2^{p+1}} \big|\widehat{f}_k\big|^2 \leqslant \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} 2^{2(\alpha-1)} \frac{\pi^{2\alpha}}{2^{2(p+1)\alpha}} = \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{2\alpha}}{2^{2(p\alpha+1)}}$$

Since for $2^{p-1} \leqslant |k| < 2^p$, $\sin^2 \dfrac{k\pi}{2^{p+1}} \geqslant \dfrac{1}{2}$, the inequality above implies that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} \big|\widehat{f}_k\big|^2 \leqslant \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{2\alpha}}{2^{2p\alpha+1}} \,.$$

4. Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0.5, 1]$. For each $p \in \mathbb{N}$, by the Cauchy inequality and the result in part 3 we obtain that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} \big|\widehat{f}_k\big| \leqslant \Big(\sum_{2^{p-1} \leqslant |k| < 2^p} 1 \Big)^{\frac{1}{2}} \Big(\sum_{2^{p-1} \leqslant |k| < 2^p} \big|\widehat{f}_k\big|^2 \Big)^{\frac{1}{2}} = \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{\alpha}}{2^{p(\alpha-\frac{1}{2})+1}} \,.$$

Therefore, by the fact that $\sum\limits_{p=1}^{\infty} \dfrac{1}{2^{p(\alpha-\frac{1}{2})}} < \infty$ (since $\alpha > \dfrac{1}{2}$), we conclude that

$$\sum_{k=-\infty}^{\infty} \big|\widehat{f}_k\big| = \big|\widehat{f}_0\big| + \sum_{p=1}^{\infty} \sum_{2^{p-1} \leqslant |k| < 2^p} \big|\widehat{f}_k\big| \leqslant \big|\widehat{f}_0\big| + \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{\alpha}}{2} \sum_{p=1}^{\infty} \frac{1}{2^{p(\alpha-\frac{1}{2})}} < \infty \,;$$

thus Problem 1 of Exercise 9 implies that the Fourier series of f converges uniformly to f on $\mathbb{R}$ if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0.5, 1]$. ◻