Exercise Problem Sets 7

Apr. 06. 2023

Problem 1. 1. Let $f : [-\pi, \pi]$ be a Riemann integrable function. Show that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = 0 \,.$$

2. Show the Riemann-Lebesgue Lemma

> If $f : [-\pi, \pi] \to \mathbb{R}$ is an integrable function, then
> $$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = 0 \,.$$

Hint: First show that for every $\varepsilon > 0$ there exists a Riemann integrable function $g : [-\pi, \pi] \to \mathbb{R}$ such that $\displaystyle\int_{-\pi}^{\pi} \big|f(x) - g(x)\big| \, dx < \varepsilon$, then apply the conclusion in 1.

Proof. 1. Let $\varepsilon > 0$ be given. Then by Lemma 6.63 in the lecture note, there exists $g \in \mathscr{C}([-\pi, \pi]; \mathbb{R})$ such that

$$f(x) \leqslant g(x) \leqslant \sup_{x\in[-\pi,\pi]} f(x) \quad \forall\, x \in [-\pi, \pi] \qquad \text{and} \qquad \int_{-\pi}^{\pi} f(x) \, dx > \int_{-\pi}^{\pi} g(x) \, dx - \frac{\varepsilon}{3} \,.$$

By the Weierstrass Theorem, there exists a polynomial p such that

$$\|g - p\|_\infty < \frac{\varepsilon}{6\pi} \,.$$

Since p is a polynomial, integrating by parts (or by Problem 5 of Exercise 6) we can show that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} p(x) \cos kx \, dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} p(x) \sin kx \, dx = 0 \,.$$

Therefore, there exists $N > 0$ such that if $k \geqslant N$,

$$\Big|\int_{-\pi}^{\pi} p(x) \cos kx \, dx\Big| < \frac{\varepsilon}{3} \qquad \text{and} \qquad \Big|\int_{-\pi}^{\pi} p(x) \sin kx \, dx\Big| < \frac{\varepsilon}{3} \,.$$

Therefore, if $k \geqslant N$,

$$\begin{aligned}
\Big|\int_{-\pi}^{\pi} f(x) \cos kx \, dx\Big| &\leqslant \Big|\int_{-\pi}^{\pi} \big[f(x) - g(x)\big] \cos kx \, dx\Big| + \Big|\int_{-\pi}^{\pi} \big[g(x) - p(x)\big] \cos kx \, dx\Big| \\
&\quad + \Big|\int_{-\pi}^{\pi} p(x) \cos kx \, dx\Big| \\
&\leqslant \int_{-\pi}^{\pi} \big|f(x) - g(x)\big| \, dx + \int_{-\pi}^{\pi} \|g - p\|_\infty \, dx + \frac{\varepsilon}{3} \\
&\leqslant \int_{-\pi}^{\pi} \big[g(x) - f(x)\big] \, dx + \int_{-\pi}^{\pi} \frac{\varepsilon}{6\pi} \, dx + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \,,
\end{aligned}$$

and similarly,

$$\Big|\int_{-\pi}^{\pi} f(x) \cos kx \, dx\Big| < \varepsilon \quad \text{whenever} \quad k \geqslant N \,.$$

2. Let $g_k(x) = (f^+ \wedge k)(x) - (f^- \wedge k)(x)$. Then

$$\begin{aligned}\int_{-\pi}^{\pi} \big|f(x) - g_k(x)\big|\,dx &= \int_{-\pi}^{\pi} \big|f^+(x) - f^-(x) - g_k(x)\big|\,dx \\ &\leqslant \int_{-\pi}^{\pi} \big|f^+(x) - (f^+ \wedge k)(x)\big|\,dx + \int_{-\pi}^{\pi} \big|f^-(x) - (f^- \wedge k)(x)\big|\,dx\,;\end{aligned}$$

thus by the fact that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} (f^+ \wedge k)(x)\,dx = \int_{-\pi}^{\pi} f^+(x)\,dx \quad \text{and} \quad \lim_{k\to\infty} \int_{-\pi}^{\pi} (f^- \wedge k)(x)\,dx = \int_{-\pi}^{\pi} f^-(x)\,dx\,,$$

we find that there exists $K > 0$ such that

$$\int_{-\pi}^{\pi} \big|f(x) - g_k(x)\big|\,dx < \frac{\varepsilon}{2} \qquad \text{whenever} \quad k \geqslant K\,.$$

Let $h = g_K$. Note that h is Riemann integrable on $[-\pi, \pi]$; thus part 1 implies that there exists $N > 0$ such that if $k \geqslant N$,

$$\Big|\int_{-\pi}^{\pi} h(x)\cos kx\,dx\Big| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \Big|\int_{-\pi}^{\pi} h(x)\sin kx\,dx\Big| < \frac{\varepsilon}{2}\,.$$

Therefore, if $k \geqslant N$,

$$\begin{aligned}\Big|\int_{-\pi}^{\pi} f(x)\cos kx\,dx\Big| &= \Big|\int_{-\pi}^{\pi} \big[f(x) - h(x)\big]\cos kx\,dx\Big| + \Big|\int_{-\pi}^{\pi} h(x)\cos kx\,dx\Big| \\ &\leqslant \int_{-\pi}^{\pi} \big|f(x) - h(x)\big|\,dx + \Big|\int_{-\pi}^{\pi} h(x)\cos kx\,dx\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and similarly,

$$\Big|\int_{-\pi}^{\pi} f(x)\sin kx\,dx\Big| < \varepsilon \qquad \text{whenever} \quad k \geqslant N\,.$$

□

Problem 2. Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$; that is, f is 2π-periodic Hölder continuous function with exponent α for some $\alpha \in (0, 1]$. Show that (without using the Berstein Theorem) the Fourier series of f converges pointwise to f, by completing the following.

1. Explain why it is enough to show that $s_n(f, 0) \to f(0)$ as $n \to \infty$. Also explain why we can assume that $f(0) = 0$.

2. Show that

$$\lim_{n\to\infty} \Big(s_n(f, 0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx\Big) = 0\,.$$

Therefore, it suffices to show that $\lim\limits_{n\to\infty} \int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx = 0$ if $f(0) = 0$.

3. Show that if $f \in \mathscr{C}^{0,\alpha}(\mathbb{R})$ and $f(0) = 0$, then the function $y = \dfrac{f(x)}{x}$ is integrable. Apply the Riemann-Lebesgue Lemma to conclude that $s_n(f, 0) \to 0$ as $n \to \infty$.

Proof. 1. Suppose that one can show that if g is a 2π-periodic Hölder continuous function with exponent $\alpha \in (0,1]$, then $s_n(g,0) \to g(0)$ as $n \to \infty$. If f is 2π-periodic Hölder continuous function with exponent $\alpha \in (0,1]$ and $a \in \mathbb{R}$, let $g(x) = f(x+a)$. Then g is a 2π-periodic Hölder continuous function with exponent α; thus $s_n(g,0) \to g(0)$ as $n \to \infty$.

On the other hand, let $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficients of f and $\{\bar{c}_k\}_{k=0}^{\infty}$ and $\{\bar{s}_k\}_{k=1}^{\infty}$ be the Fourier coefficients of g. Then

$$\begin{aligned}\bar{c}_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+a)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos k(x-a)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\big(\cos kx\cos ka + \sin kx \sin ka\big)\,dx \\ &= c_k\cos ka + s_k \sin ka\,.\end{aligned}$$

Note that

$$s_n(g,0) = \frac{\bar{c}_0}{2} + \sum_{k=1}^{n}\big[\bar{c}_k\cos(k\cdot 0) + \bar{s}_k\sin(k\cdot 0)\big] = \sum_{k=1}^{n}\big(c_k\cos ka + s_k\sin ka\big) = s_n(f,a)\,;$$

thus the fact that $g(0) = f(a)$ implies that $s_n(f,a) \to f(a)$ as $n \to \infty$. Moreover, if $f(0) \neq 0$, we consider the function $h(x) = f(x) - f(0)$. Then $h(0) = 0$ and $s_n(f,x) = s_n(h,x) + f(0)$ so that if the $s_n(h,0)$ converges to 0, then $s_n(f,0)$ converges to $f(0)$. In other words, we can further assume that $f(0) = 0$.

2. Note that $s_n(f,x) = (D_n \star f)(x)$; thus

$$s_n(f,0) = \int_{-\pi}^{\pi} f(x)\frac{\sin(n+\frac{1}{2})x}{2\pi\sin\frac{x}{2}}\,dx\,.$$

Therefore,

$$\begin{aligned}s_n(f,0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big[\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} - \frac{\sin nx}{x}\Big]\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big(\frac{\sin nx\cos\frac{x}{2} + \sin\frac{x}{2}\cos nx}{2\sin\frac{x}{2}} - \frac{\sin nx}{x}\Big)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)\sin nx\,dx\,.\end{aligned}$$

Note that

$$\lim_{x\to 0}\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big) = \lim_{x\to 0}\frac{x\cos\frac{x}{2} - 2\sin\frac{x}{2}}{2x\sin\frac{x}{2}} = \lim_{x\to 0}\frac{x(1-\frac{x^2}{8}) - 2(\frac{x}{2} - \frac{x^3}{48})}{2x\cdot\frac{x}{2}} = 0\,;$$

thus the function $y = f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)$ is continuous on $[-\pi,\pi]$. By the Riemann-Lebesgue Lemma,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)\sin nx\,dx = 0\,.$$

Therefore,

$$\lim_{n\to\infty}\Big(s_n(f,0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx\Big) = 0\,.$$

3. Since $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0,1]$,

$$M \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty\,.$$

In particular, if $x \neq 0$,

$$\frac{|f(x)|}{|x|^\alpha} = \frac{|f(x) - f(0)|}{|x-0|^\alpha} \leqslant \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = M < \infty\,;$$

thus

$$\left|\frac{f(x)}{x}\right| \leqslant M|x|^{\alpha-1} \qquad \forall\, x \neq 0\,.$$

Therefore, the comparison test implies that the function $y = \dfrac{f(x)}{x}$ is integrable on $[-\pi, \pi]$ since

$$\int_0^\pi x^{\alpha-1}\,dx = \lim_{\varepsilon \to 0^+} \frac{1}{\alpha} x^\alpha \Big|_{x=\varepsilon}^{\pi} = \frac{\pi^\alpha}{\alpha}$$

and the change of varialbe $x \mapsto -x$ shows that

$$\int_{-\pi}^0 |x|^{\alpha-1}\,dx = \int_0^\pi x^{\alpha-1}\,dx = \frac{\pi^\alpha}{\alpha}\,.$$

The Riemann-Lebesgue Lemma then implies that $\displaystyle\lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{f(x)}{x} \sin nx\,dx = 0$. □

Problem 3. Assuming that the improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = I$ exists. Establish its value by first using the Riemann-Lebesgue lemma to show that

$$I = \lim_{n\to\infty} \int_0^{n\pi} \frac{\sin x}{x}\,dx = \frac{\pi}{2} \lim_{n\to\infty} \int_{-\pi}^{\pi} D_n(x)\,dx\,,$$

where D_n is the Dirchlet kernel.

Proof. By the substitution of variable $x \to nx$,

$$\int_0^{n\pi} \frac{\sin x}{x}\,dx = \int_0^\pi \frac{\sin nx}{x}\,dx\,.$$

Similar to the proof of Problem 2, we have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left[\frac{\sin nx}{x} - \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}\right] dx = 0\,;$$

thus by the fact that the function $x \mapsto \dfrac{\sin nx}{x}$ is even, using the definition of the Dirichlet kernel we find that

$$I = \lim_{n\to\infty} \int_0^{n\pi} \frac{\sin x}{x}\,dx = \frac{1}{2} \lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{\sin nx}{x}\,dx = \frac{1}{2} \lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}\,dx = \frac{\pi}{2} \lim_{n\to\infty} \int_{-\pi}^{\pi} D_n(x)\,dx\,.$$

Since $\displaystyle\int_{-\pi}^{\pi} D_n(x)\,dx = 1$, we conclude that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. □

Problem 4. Let f be the 2π-periodic function defined by $f(x) = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$ for $|x| \leqslant \pi$. Express it as a Fourier series and compute

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1}\,.$$

Proof. Note that f is an even function, so the Fourier coefficients $\{s_k\}_{k=1}^{\infty}$ associated with f is the zero sequence. Moreover, using the formula

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C\,,$$

and we find that

$$\begin{aligned} c_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{2\pi}\frac{e^x(\cos kx + k\sin kx) + e^{-x}(-\cos kx + k\sin kx)}{1+k^2}\Big|_{x=-\pi}^{x=\pi} \\ &= \frac{e^{\pi}(-1)^k + e^{-\pi}(-1)^{k+1} - e^{-\pi}(-1)^k + e^{\pi}(-1)^k}{2\pi(1+k^2)} = \frac{e^{\pi}-e^{-\pi}}{\pi}\frac{(-1)^k}{k^2+1}\,. \end{aligned}$$

Since $f \in \mathscr{C}^{0,1}(\mathbb{T})$, we find that

$$\frac{e^x + e^{-x}}{2} = \frac{e^{\pi}-e^{-\pi}}{2\pi} + \frac{e^{\pi}-e^{-\pi}}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\cos kx}{k^2+1} \qquad \forall\, x \in [-\pi,\pi]\,.$$

Therefore,

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{e^{\pi}-e^{-\pi}}{2\pi} + \frac{e^{\pi}-e^{-\pi}}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2+1}\,.$$

which shows that

$$\sum_{k=0}^{\infty}\frac{1}{k^2+1} = 1 + \frac{\pi}{e^{\pi}-e^{-\pi}}\Big(\frac{e^{\pi}+e^{-\pi}}{2} - \frac{e^{\pi}-e^{-\pi}}{2\pi}\Big) = \frac{1}{2} + \frac{\pi(e^{\pi}+e^{-\pi})}{2(e^{\pi}-e^{-\pi})}\,.$$ □

Problem 5. Let f be the 2π-periodic function defined by $f(x) = \cos(ax)$ for $|x| \leqslant \pi$, where a is not an integer. Express it as a Fourier series and deduce the identity

$$\pi\cot(\pi a) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{a+n} \qquad \forall\, a \neq \mathbb{Z}\,.$$

Proof. First we note that f is an even function, so the Fourier coefficients $\{s_k\}_{k=1}^{\infty}$ is the zero sequence. On the other hand, the Fourier coefficients $\{c_k\}_{k=1}^{\infty}$ are given by

$$\begin{aligned} c_k &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(ax)\cos(kx)\,dx = \frac{1}{2\pi}\Big[\frac{\sin(k+a)x}{k+a} + \frac{\sin(k-a)x}{k-a}\Big]\Big|_{x=-\pi}^{x=\pi} \\ &= \frac{1}{\pi}\Big[\frac{\sin(k+a)\pi}{k+a} + \frac{\sin(k-a)\pi}{k-a}\Big] = \frac{(-1)^k}{\pi}\Big(\frac{1}{k+a} - \frac{1}{k-a}\Big)\sin(a\pi)\,. \end{aligned}$$

Therefore, by the fact that $f \in \mathscr{C}^{0,1}(\mathbb{T})$,

$$f(x) = \cos(ax) = \frac{\sin(a\pi)}{a\pi} + \frac{\sin(a\pi)}{\pi}\sum_{k=1}^{\infty}(-1)^k\Big(\frac{1}{k+a} - \frac{1}{k-a}\Big)\cos kx \qquad \forall\, x \in [-\pi,\pi]\,.$$

In particular, evaluating the function at $x = \pi$ we obtain that

$$\begin{aligned}\cos(a\pi) &= \frac{\sin(a\pi)}{a\pi} + \frac{\sin(a\pi)}{\pi}\sum_{k=1}^{\infty}(-1)^k\Big(\frac{1}{k+a} - \frac{1}{k-a}\Big)\cos k\pi \\ &= \frac{\sin(a\pi)}{a\pi} + \frac{\sin(a\pi)}{\pi}\sum_{k=1}^{\infty}\Big(\frac{1}{k+a} - \frac{1}{k-a}\Big)\end{aligned}$$

thus

$$\pi\cot(a\pi) = \frac{1}{a} + \sum_{k=1}^{\infty}\Big(\frac{1}{k+a} - \frac{1}{k-a}\Big).$$

On the other hand,

$$\begin{aligned}\sum_{n=-N}^{N}\frac{1}{a+n} &= \sum_{n=-N}^{-1}\frac{1}{a+n} + \frac{1}{a} + \sum_{n=1}^{N}\frac{1}{a+n} = \sum_{n=1}^{N}\frac{1}{a-n} + \frac{1}{a} + \sum_{n=1}^{N}\frac{1}{a+n} \\ &= \frac{1}{a} + \sum_{n=1}^{N}\Big(\frac{1}{a+n} - \frac{1}{n-a}\Big),\end{aligned}$$

so passing to the limit as $N \to \infty$ we conclude that desired result. $\square$

Problem 6. A family of functions $\big\{\varphi_n \in \mathscr{C}(\mathbb{T}) \,\big|\, n \in \mathbb{N}\big\}$ is called an ***approximation of the identity*** if

(1) $\varphi_n(x) \geqslant 0$; (2) $\displaystyle\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$ for every $n \in \mathbb{N}$;

(3) $\displaystyle\lim_{n\to\infty}\int_{\delta\leqslant|x|\leqslant\pi} \varphi_n(x)\,dx = 0$ for every $\delta > 0$, here we identify $\mathbb{T}$ with the interval $[-\pi, \pi]$.

Show that if $\{\varphi_n\}_{n=1}^{\infty}$ is an approximation of the identity and $f \in \mathscr{C}(\mathbb{T})$, then $\{\varphi_n \star f\}_{n=1}^{\infty}$ converges uniformly to f as $n \to \infty$.

Proof. W.L.O.G., we may assume that $f \not\equiv 0$. By the definition of the convolution,

$$\begin{aligned}\big|(\varphi_n \star f)(x) - f(x)\big| &= \int_{\mathbb{T}} \varphi_n(x-y)f(y)\,dy - f(x) \\ &= \int_{\mathbb{T}} \varphi_n(x-y)\big(f(x) - f(y)\big)dy\,,\end{aligned}$$

where we use (2) of the definition above to obtain the last equality. Now given $\varepsilon > 0$. Since $f \in \mathscr{C}(\mathbb{T})$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$ whenever $|x - y| < \delta$. Therefore,

$$\begin{aligned}&|(\varphi_n \star f)(x) - f(x)| \\ &\quad\leqslant \int_{|x-y|<\delta} \varphi_n(x-y)\big|f(x) - f(y)\big|dy + \int_{\delta\leqslant|x-y|} \varphi_n(x-y)\big|f(x) - f(y)\big|dy \\ &\quad\leqslant \frac{\varepsilon}{2}\int_{\mathbb{T}} \varphi_n(x-y)\,dy + 2\max_{\mathbb{T}}|f|\int_{\delta\leqslant|z|\leqslant\pi} \varphi_n(z)\,dz\,.\end{aligned}$$

By (3) of the definition above, there exists $N > 0$ such that if $n \geqslant N$,

$$\int_{\delta\leqslant|z|\leqslant\pi} \varphi_n(z)\,dx < \frac{\varepsilon}{4\max_{\mathbb{T}}|f|}\,.$$

Therefore, for $n \geqslant N$, $\big|(\varphi_n \star f)(x) - f(x)\big| < \varepsilon$ for all $x \in \mathbb{T}$. $\square$

Problem 7. In this problem we show that the collection of trigonometric polynomials $\mathscr{P}(\mathbb{T})$ (defined in Corollary 7.85 in the lecture note) is dense in $\mathscr{C}(\mathbb{T})$ in another way. Complete the following.

1. Let $\varphi_n(x) = c_n(1+\cos x)^n$, where c_n is chosen so that $\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$. Show that
$$c_n = \frac{2^{n-1}}{\pi}\frac{(n!)^2}{(2n)!}\,.$$

2. Show that for each $0 < \delta < \pi$,
$$\lim_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx = 0\,.$$
In other words, $\{\varphi_n\}_{n=1}^\infty$ is an approximation of the identity. Therefore, Problem 6 shows that $\{\varphi_n \star f\}_{n=1}^\infty$ converges uniformly to f as $n \to \infty$ if $f \in \mathscr{C}(\mathbb{T})$.

3. Show that $\mathscr{P}(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$.

Proof. 1. Let $\varphi_n(x) = c_n(1+\cos x)^n$, where c_n is chosen so that $\int_{\mathbb{T}} \varphi_n(x)\,dx = 1$. First we note that by Wallis's formula,
$$\int_{-\pi}^{\pi} (1+\cos x)^n\,dx = 2^n \int_{-\pi}^{\pi} \Big(\frac{1+\cos x}{2}\Big)^n dx = 2^n \int_{-\pi}^{\pi} \cos^{2n}\frac{x}{2}\,dx = 2^{n+1}\int_0^{\pi} \cos^{2n}\frac{x}{2}\,dx$$
$$= 2^{n+2}\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = 2^{n+2}\frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2} = \frac{\pi(2n)!}{2^{n-1}(n!)^2}\,.$$
Therefore,
$$1 = \int_{\mathbb{T}} \varphi_n(x)\,dx = c_n \int_{-\pi}^{\pi} (1+\cos x)^n\,dx = \frac{\pi(2n)!}{2^{n-1}(n!)^2} c_n$$
which implies that
$$c_n = \frac{2^{n-1}}{\pi}\frac{(n!)^2}{(2n)!}\,.$$

2. Now $\{\varphi_n\}_{n=1}^\infty$ is clearly non-negative and satisfies (2) of Definition of an approximation of the identity (given in Problem 6) for all $n \in \mathbb{N}$. Let $\delta > 0$ be given.
$$\int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx \leqslant \int_{\delta \leqslant |x| \leqslant \pi} c_n(1+\cos\delta)^n dx \leqslant 2^{2n}\Big(\frac{1+\cos\delta}{2}\Big)^n \frac{(n!)^2}{(2n)!}\,.$$
By Stirling's formula $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}n^n e^{-n}} = 1$,
$$\limsup_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx \leqslant \limsup_{n\to\infty} 2^{2n}\Big(\frac{1+\cos\delta}{2}\Big)^n \frac{\big(\sqrt{2\pi n}n^n e^{-n}\big)^2}{\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}}$$
$$= \limsup_{n\to\infty} \sqrt{\pi n}\Big(\frac{1+\cos\delta}{2}\Big)^n = 0\,;$$
thus
$$\lim_{n\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} \varphi_n(x)\,dx = 0\,.$$
So $\{\varphi_n\}_{n=1}^\infty$ is an approximation of the identity. By the result in Problem 6, $\varphi_k \star f$ converges uniformly to f if $f \in \mathscr{C}(\mathbb{T})$.

3. To conclude part 3, we note that for each $n \in \mathbb{N}$, $\varphi_n \star f$ is a trigonometric function. Therefore, part 2 implies that any function in $\mathscr{C}(\mathbb{T})$ can be approximated by trigonometric functions; thus $\mathscr{P}(\mathbb{T})$ is dense in $\mathscr{C}(\mathbb{T})$. $\square$