Exercise Problem Sets 6

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Problem 1. Let $\delta : (\mathscr{C}([-1,1];\mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is linear and uniformly continuous.

Proof. Let $c \in \mathbb{R}$ and $f, g \in \mathscr{C}([-1,1];\mathbb{R})$. Then

$$\delta(cf + g) = cf(0) + g(0) = c\delta(f) + \delta(g)$$

which shows that δ is linear on $\mathscr{C}([-1,1];\mathbb{R})$.

For the uniform continuity of δ, let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then if $\|f - g\|_\infty < \delta$, we have

$$\big|f(0) - g(0)\big| \leqslant \|f - g\|_\infty < \delta = \varepsilon$$

which implies that δ is uniformly continuous. □

Problem 2. Let (M, d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a < f(x) < b \text{ for all } x \in K\big\}$ is open in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \big\{f \in \mathscr{C}(K;\mathbb{R}) \,\big|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\big\}$ is closed in $\big(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty\big)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \big\{f \in \mathscr{C}_b(A;\mathbb{R}) \,\big|\, f(x) > 0 \text{ for all } x \in A\big\}$ is open in $\big(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty\big)$.

Proof. 1. Let $g \in U$. By the Extreme Value Theorem (Corollary **??**), there exists $x_0, x_1 \in K$ such that

$$g(x_0) = \inf_{x \in K} g(x) \qquad \text{and} \qquad g(x_1) = \sup_{x \in K} g(x)\,.$$

Therefore, $a < \inf\limits_{x \in K} g(x) \leqslant \sup\limits_{x \in K} g(x) < b$. Let $r = \min\big\{b - \sup\limits_{x \in K} g(x), \inf\limits_{x \in K} g(x) - a\big\}$. Then $r > 0$. Moreover, if $f \in B(g, r)$ and $x \in K$, we have

$$|f(x) - g(x)| \leqslant \sup_{x \in K} \big|f(x) - g(x)\big| = \|f - g\|_\infty < r\,.$$

Therefore, if $f \in B(g, r)$, by the fact that $r \leqslant b - \sup\limits_{x \in K} g(x)$ and $r \leqslant \inf\limits_{x \in K} g(x) - a$, we conclude that if $x \in K$,

$$a \leqslant \inf_{x \in K} g(x) - r \leqslant g(x) - r < f(x) < g(x) + r \leqslant \sup_{x \in K} g(x) + r \leqslant b$$

which implies that $f \in U$. Therefore, $B(g, r) \subseteq U$; thus U is open.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in F such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on K. Then $f \in \mathscr{C}(K;\mathbb{R})$. Moreover, by the fact that $a \leqslant f_n(x) \leqslant b$ for all $x \in K$ and $n \in \mathbb{N}$, we find that $a \leqslant f(x) \leqslant b$ for all $x \in K$ since $f(x) = \lim\limits_{n\to\infty} f_n(x)$. This implies that $f \in F$; thus F is closed (since it contains all the limit points).

3. Consider the case $A = (0,1)$. Then the function $f(x) = x$ belongs to B; however, for every $r > 0$, the function $g(x) = f(x) - \frac{r}{2}$ belongs to $B(f,r)$ since

$$\|f - g\|_\infty = \sup_{x\in(0,1)} \big|f(x) - g(x)\big| = \frac{r}{2} < r\,.$$

However, $g \notin B$ since if $0 < x \ll 1$, we have $g(x) < 0$. In other words, there exists no $r > 0$ such that $B(f,r) \subseteq B$; thus B is not open. □

Problem 3. Define B to be the set of all even functions in the space $\mathscr{C}([-1,1];\mathbb{R})$; that is, $f \in B$ if and only if f is continuous on $[-1,1]$ and $f(x) = f(-x)$ for all $x \in [-1,1]$. Prove that B is closed but not dense in $\mathscr{C}([-1,1];\mathbb{R})$. Hence show that even polynomials are dense in B, but not in $\mathscr{C}([-1,1];\mathbb{R})$.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in B and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $[-1,1]$. Then f is continuous. Moreover, for each $x \in [-1,1]$,

$$f(x) = \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} f_k(-x) = f(-x)\,;$$

thus f is even. Therefore, $f \in B$ which shows that B is closed. However, B is not dense in B since there exists no $f \in B$ satisfying that

$$\max_{x\in[-1,1]} \big|f(x) - x\big| < \frac{1}{2}$$

since

$$\max_{x\in[-1,1]} \big|f(x) - x\big| \geqslant \max\{\big|f(1) - 1\big|, \big|f(-1) + 1\big|\} = \max\{\big|f(1) - 1\big|, \big|f(1) + 1\big|\} \geqslant 1\,.$$

Let $\mathcal{A}$ denote the collection of even polynomials, and f be an even continuous function. Then the Weierstrass Theorem implies that there exists a sequence of polynomial $\{p_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \max_{x\in[0,1]} \big|f(\sqrt{x}) - p_n(x)\big| = 0\,.$$

For each $n \in \mathbb{N}$, define $q_n : [-1,1] \to \mathbb{R}$ by $q_n(x) = p_n(x^2)$. Then $\{q_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ and

$$\lim_{n\to\infty} \max_{x\in[-1,1]} \big|f(x) - q_n(x)\big| = \lim_{n\to\infty} \max_{x\in[0,1]} \big|f(x) - p_n(x^2)\big| = \lim_{n\to\infty} \max_{x\in[0,1]} \big|f(\sqrt{x}) - p_n(x)\big| = 0$$

which shows that $\{q_n\}_{n=1}^{\infty}$ converges uniformly to f on $[-1,1]$; thus $\mathcal{A}$ is dense in B. On the other hand, since $\mathcal{A} \subseteq B$, we must have $\bar{\mathcal{A}} \subseteq \bar{B} \subsetneq \mathscr{C}([-1,1];\mathbb{R})$ which implies that $\mathcal{A}$ is not dense in $\mathscr{C}([-1,1];\mathbb{R})$. □

Problem 4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function.

1. Suppose that
$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,.$$
Show that $f = 0$ on $[0,1]$.

2. Suppose that for some $m \in \mathbb{N}$,
$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \{0,1,\cdots,m\}\,.$$
Show that $f(x) = 0$ has at least $(m+1)$ distinct real roots around which $f(x)$ change signs.

Proof. 1. By the Weierstrass Theorem, for each $k \in \mathbb{N}$ there exists a polynomial p_k such that $\|f - p_k\|_\infty < \dfrac{1}{k}$. Since $\displaystyle\int_0^1 f(x)x^n\, dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$, we find that
$$\int_0^1 f(x)p_k(x)\, dx = 0 \qquad \forall\, k \in \mathbb{N}\,.$$
Note that $f(f - p_k)$ converges to the zero function uniformly on $[0,1]$ since
$$\|f(f-p_k)\|_\infty \leqslant \|f\|_\infty \|f - p_k\|_\infty \leqslant \frac{1}{k}\|f\|_\infty \to 0 \;\text{ as }\; k \to \infty\,;$$
thus by the fact that
$$\int_0^1 f(x)^2\, dx = \int_0^1 f(x)\big[f(x) - p_k(x)\big]\, dx\,,$$
we find that $\displaystyle\int_0^1 f(x)^2\, dx = 0$. Therefore, by the continuity of f, we conclude that $f = 0$ on $[0,1]$.

2. Let
$$D = \Big\{k \in \mathbb{N} \,\Big|\, \text{if } f \in \mathscr{C}([0,1];\mathbb{R}) \text{ and } f \text{ changes signs around } 0 < \alpha_1 < \cdots < \alpha_k < 1, \text{ then } y = f(x)\prod_{j=1}^{k}(x - \alpha_j) \text{ does not change sign}\Big\}\,.$$
Suppose that $f \in \mathscr{C}([0,1];\mathbb{R})$ changes sign only around $0 < \alpha_1 < 1$. Then $y = f(x)(x - \alpha_1)$ does not change sign so that $1 \in D$. Assume that $k \in D$. If f changes signs only around $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{k+1} < 1$, then the function $y = f(x)(x - \alpha_{k+1})$ changes signs only around $0 < \alpha_1 < \cdots < \alpha_k < 1$; thus $y = f(x)(x - \alpha_{k+1})\prod\limits_{j=1}^{k}(x - \alpha_j) = f(x)\prod\limits_{j=1}^{k+1}(x - \alpha_j)$ does not change sign which shows that $k + 1 \in D$. By induction, we conclude that $D = \mathbb{N}$.

Now suppose the contrary that $f(x) = 0$ has at most m distinct real roots $0 < \alpha_1 < \cdots < \alpha_k < 1$, where $0 \leqslant k \leqslant m$, around which $f(x)$ change signs. Then $y = f(x)\prod\limits_{j=1}^{k}(x - \alpha_j)$ does not change sign. W.L.O.G., we assume that $f(x)\prod_{j=1}^{k}(x-\alpha_j) \geqslant 0$ for all $x \in [0,1]$. Then by the fact that

$$\int_0^1 f(x)x^n dx = 0 \qquad \forall\, n \in \{0,1,\cdots,m\}\,.$$

and $k \leqslant m$, we find that

$$\int_0^1 f(x)\prod_{j=1}^{k}(x-\alpha_j)\,dx = 0\,;$$

thus the sign-definite property and the continuity of the function $y = f(x)\prod_{j=1}^{k}(x-\alpha_j)$ implies that $f(x)\prod_{j=1}^{k}(x-\alpha_j) = 0$ for all $x \in [0,1]$. Therefore, $f(x)\prod_{j=1}^{k}(x-\alpha_j) = 0$ for all $x \in [0,1]\backslash\{\alpha_1,\alpha_2,\cdots,\alpha_k\}$ or equivalently, $f(x) = 0$ for all $x \in [0,1]\backslash\{\alpha_1,\alpha_2,\cdots,\alpha_k\}$. The continuity of f further implies that $f = 0$ on $[0,1]$, a contradiction to that f has at most m distinct real roots around which f changes signs. □

Problem 5. Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\lim_{n\to\infty}\int_0^1 f(x)\cos(nx)\,dx = 0 \qquad \text{and} \qquad \lim_{n\to\infty}\int_0^1 f(x)\sin(nx)\,dx = 0\,.$$

Proof. We only show the latter case since the proof of the former case is the same.

We first show that $\lim\limits_{n\to\infty}\int_0^1 x^k\sin(nx)\,dx = 0$ for all $k \in \mathbb{N}\cup\{0\}$. Let

$$D = \Big\{k \in \mathbb{N}\cup\{0\}\,\Big|\, \lim_{n\to\infty}\int_0^1 x^k\sin(nx)\,dx = 0\Big\}\,.$$

Then $0 \in D$ and $1 \in D$ since

$$\int_0^1 \sin(nx)\,dx = \frac{-\cos(nx)}{n}\Big|_{x=0}^{x=1} = \frac{\cos 0 - \cos n}{n} \to 0 \ \text{ as } \ n \to \infty$$

and

$$\int_0^1 x\sin(nx)\,dx = \frac{-x\cos(nx)}{n}\Big|_{x=0}^{x=1} + \frac{1}{n}\int_0^1 \cos(nx)\,dx = -\frac{\cos n}{n} + \frac{\sin n}{n^2} \to 0 \ \text{ as } \ n \to \infty\,.$$

Suppose that $\{0,1,\cdots,k\} \subseteq D$. Then

$$\begin{aligned}\int_0^1 x^{k+1}\sin(nx)\,dx &= -\frac{x^{k+1}\cos(nx)}{n}\Big|_{x=0}^{x=1} + \frac{k+1}{n}\int_0^1 x^k\cos(nx)\,dx \\ &= -\frac{\cos n}{n} + \frac{k+1}{n}\Big[\frac{x^k\sin(nx)}{n}\Big|_{x=0}^{x=1} - \frac{k}{n}\int_0^1 x^{k-1}\sin(nx)\,dx\Big] \\ &= -\frac{\cos n}{n} + \frac{(k+1)\sin n}{n^2} - \frac{(k+1)k}{n^2}\int_0^1 x^{k-1}\sin(nx)\,dx \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

By induction, $D = \mathbb{N}\cup\{0\}$.

Having established that $D = \mathbb{N}\cup\{0\}$, we immediately conclude that

$$\lim_{n\to\infty}\int_0^1 p(x)\sin(nx)\,dx = 0 \qquad \text{for all polynomial } p\,.$$

Let $\varepsilon > 0$ be given. By the Weierstrass Theorem, there exists a polynomial p such that $\|f-p\|_\infty < \dfrac{\varepsilon}{2}$. By the fact that $\lim\limits_{n\to\infty}\displaystyle\int_0^1 p(x)\sin(nx)\,dx = 0$, there exists $N > 0$ such that

$$\Big|\int_0^1 p(x)\sin(nx)\,dx\Big| < \frac{\varepsilon}{2} \quad \textit{whenever} \quad n \geqslant N\,.$$

Therefore, if $n \geqslant N$,

$$\begin{aligned}\Big|\int_0^1 f(x)\sin(nx)\,dx\Big| &\leqslant \Big|\int_0^1 \big[f(x)-p(x)\big]\sin(nx)\,dx\Big| + \Big|\int_0^1 p(x)\sin(nx)\,dx\Big| \\ &\leqslant \int_0^1 \|f-p\|_\infty\,dx + \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

which establishes that $\lim\limits_{n\to\infty}\displaystyle\int_0^1 f(x)\sin(nx)\,dx = 0$. □

Problem 6. Put $p_0 = 0$ and define

$$p_{k+1}(x) = p_k(x) + \frac{x^2 - p_k^2(x)}{2} \qquad \forall\, k \in \mathbb{N}\cup\{0\}\,.$$

Show that $\{p_k\}_{k=1}^\infty$ converges uniformly to $|x|$ on $[-1,1]$.

Hint: Use the identity

$$|x| - p_{k+1}(x) = \big[|x| - p_k(x)\big]\Big[1 - \frac{|x| + p_k(x)}{2}\Big] \tag{0.1}$$

to prove that $0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|$ if $|x| \leqslant 1$, and that

$$|x| - p_k(x) \leqslant |x|\Big(1 - \frac{|x|}{2}\Big)^k < \frac{2}{k+1}$$

if $|x| \leqslant 1$.

Proof. Let $D = \big\{k\in\mathbb{N}\,\big|\,0 \leqslant p_k(x) \leqslant p_{k+1}(x) \leqslant |x|\ \forall\, x\in[-1,1]\big\}$. We first note that if $0 \leqslant p_k(x) \leqslant |x|$ for all $x \in [-1,1]$, then

1. using the iterative formula, $p_{k+1}(x) - p_k(x) = \dfrac{x^2 - p_k^2(x)}{2} \geqslant 0$ for all $x \in [-1,1]$ which shows that $p_{k+1}(x) \geqslant p_k(x) \geqslant 0$.

2. using $(\star)$ we find that $|x| - p_{k+1}(x) \geqslant \big[|x| - p_k(x)\big](1-|x|) \geqslant 0$ which shows that $p_{k+1}(x) \leqslant |x|$.

Therefore, D is indeed the set $\big\{k\in\mathbb{N}\,\big|\,0 \leqslant p_k(x) \leqslant |x|\ \forall\, x\in[-1,1]\big\}$. The fact that $p_1(x) = \dfrac{x^2}{2}$ implies that $1 \in D$, while if $k \in D$ implies that $k+1 \in D$. By induction, $D = \mathbb{N}$.

Using $(\star)$ again, we find that

$$0 \leqslant |x| - p_k(x) = \big[|x| - p_{k-1}(x)\big]\Big[1 - \frac{|x|+p_k(x)}{2}\Big] \leqslant \big[|x| - p_{k-1}(x)\big]\big(1 - \frac{|x|}{2}\big) \qquad \forall\, k\in\mathbb{N}\,;$$

thus

$$\begin{aligned}0 \leqslant |x| - p_k(x) &\leqslant \big[|x| - p_{k-1}(x)\big]\big(1 - \frac{|x|}{2}\big) \leqslant \big[|x| - p_{k-2}(x)\big]\big(1 - \frac{|x|}{2}\big) \\ &\leqslant \cdots \leqslant \big[|x| - p_0(x)\big]\big(1 - \frac{|x|}{2}\big)^k = |x|\big(1 - \frac{|x|}{2}\big)^k\,.\end{aligned}$$

By the fact that $|x|\big(1-\frac{|x|}{2}\big)^k \leqslant \frac{2}{k+1}$ for all $x\in[-1,1]$, we conclude that

$$\lim_{k\to\infty}\max_{x\in[-1,1]}\big|p_k(x)-|x|\big|=0$$

which shows that $\{p_k\}_{k=1}^\infty$ converges uniformly to $y=|x|$ on $[-1,1]$. □

Problem 7. Let $f:[0,1]\to\mathbb{R}$ be continuous and $\varepsilon>0$. Prove that there is a simple function g (defined in Example 7.75 in the lecture note) such that $\|f-g\|_\infty<\varepsilon$.

Proof. Let $\varepsilon>0$ be given. Since f is continuous on $[0,1]$, f is uniformly continuous; thus there exists $\delta>0$ such that

$$\big|f(x)-f(y)\big|<\frac{\varepsilon}{2}\qquad\text{whenever}\quad |x-y|<\delta \text{ and } x,y\in[0,1]\,.$$

Let $n>0$ be such that $\frac{1}{n}<\delta$, and let $x_k=\frac{k}{n}$ for $0\leqslant k\leqslant n$. Then $\mathcal{P}=\big\{0=x_0<x_1<\cdots<x_n=1\big\}$ is a partition of $[0,1]$. Define

$$g(x)=\left\{\begin{array}{ll} g(x_k) & \text{if } x\in[x_k,x_{k+1}) \text{ and } 0\leqslant k\leqslant n-2\,,\\ g(x_{n-1}) & \text{if } x\in[x_{n-1},x_n]\,.\end{array}\right.$$

Then g is a simple function, and $\big|f(x)-g(x)\big|<\varepsilon$ for all $x\in[0,1]$. The latter implies that

$$\|f-g\|_\infty\equiv\sup_{x\in[0,1]}\big|f(x)-g(x)\big|\leqslant\frac{\varepsilon}{2}<\varepsilon$$

which shows that we find out function g. □

Problem 8. Suppose that p_n is a sequence of polynomials converging uniformly to f on $[0,1]$ and f is not a polynomial. Prove that the degrees of p_n are not bounded.
Hint: An Nth-degree polynomial p is uniquely determined by its values at $N+1$ points $x_0,\cdots,x_N$ via Lagrange's interpolation formula

$$p(x)=\sum_{k=0}^N\pi_k(x)\frac{p(x_k)}{\pi_k(x_k)}\,,$$

where $\pi_k(x)=(x-x_0)(x-x_1)\cdots(x-x_N)/(x-x_k)=\prod_{\substack{1\leqslant j\leqslant N\\ j\neq k}}(x-x_j)$.

Proof. Suppose the contrary that there exists a sequence of polynomial $\{p_n\}_{k=1}^\infty$ which converges uniformly to f on $[0,1]$ and $\deg(p_n)\leqslant N$ for all $n\in\mathbb{N}$. W.L.O.G. we assume that

$$\|p_n-f\|_\infty<1\qquad\forall\, n\in\mathbb{N}\,.$$

Then $\big|p_n(x)\big|\leqslant\|f\|_\infty+1$ for all $x\in[0,1]$ and $n\in\mathbb{N}$.

Since $\deg(p_n)\leqslant N$, using the Lagrange interpolation formula with $x_k=k/N$, we have

$$p_n(x)=\sum_{k=0}^N\pi_k(x)\frac{p_n(x_k)}{\pi_k(x_k)}=\sum_{j=0}^N a_{jn}x^j\,.$$

Let $[N/2]$ denote the largest integer smaller than $N/2$. Note that

$$\big|\pi_k(x_k)\big| = \frac{k}{N}\cdot\frac{k-1}{N}\cdot\dots\cdot\frac{1}{N}\cdot\frac{1}{N}\cdot\dots\cdot\frac{N-k}{N} \geqslant \frac{[N/2]!}{N^N}$$

so that

$$\Big|\frac{p_n(x_k)}{\pi_k(x_k)}\Big| \leqslant \frac{(\|f\|_\infty + 1)N^N}{[N/2]!}\,.$$

Moreover, $\pi_k(x) = \sum\limits_{j=0}^{N} c_j x^j$ with $|c_j| \leqslant C^N_{[N/2]}$. Therefore,

$$|a_{jn}| = \left|\sum_{k=0}^{N} c_j \frac{p_n(x_k)}{\pi_k(x_k)}\right| \leqslant (N+1)\frac{(\|f\|_\infty + 1)N^N}{[N/2]!}C^N_{[N/2]} \qquad \forall\, 0 \leqslant j \leqslant N \text{ and } n \in \mathbb{N}\,.$$

In other words, the coefficients of each p_n is bounded by a fixed constant. This allows us to pick a subsequence $\{p_{n_k}\}_{k=1}^\infty$ of $\{p_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty} a_{jn_k} = a_j \text{ exists for all } 0 \leqslant j \leqslant N\,.$$

This implies that $\{p_{n_k}\}_{k=1}^\infty$ converges uniformly to the polynomial $p(x) = \sum\limits_{j=0}^{N} a_j x^j$ since $\{p_{n_k}\}_{k=1}^\infty$ converges pointwise to p and $\{p_n\}_{n=1}^\infty$ converges uniformly on $[0,1]$ so that $\{p_{n_k}\}_{k=1}^\infty$ converges uniformly on $[0,1]$. On the other hand, since $\{p_n\}_{n=1}^\infty$ converges uniformly to f on $[0,1]$, we conclude that $f = p$, a contradiction. □

Problem 9. Consider the set of all functions on $[0,1]$ of the form

$$h(x) = \sum_{j=1}^{n} a_j e^{b_j x}\,,$$

where $a_j,\ b_j \in \mathbb{R}$. Is this set dense in $\mathscr{C}([0,1];\mathbb{R})$?

Proof. Let $\mathcal{A} = \Big\{ \sum_{j=1}^n a_j e^{b_j x} \,\Big|\, a_j, b_j \in \mathbb{R}\Big\}$. Then

1. $\mathcal{A}$ is an algebra since if $f(x) = \sum\limits_{j=1}^{n} a_j e^{b_j x}$ and $g(x) = \sum\limits_{k=1}^{m} c_k e^{d_k x}$, we have

   $$\Big(\sum_{j=1}^{n} a_j e^{b_j x}\Big)\Big(\sum_{k=1}^{m} c_k e^{d_k x}\Big) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j c_k e^{(b_j+d_k)x} = \sum_{\ell=1}^{N} A_\ell e^{B_\ell x}$$

   for some $A_\ell, B_\ell \in \mathbb{R}$, and clearly, $f + g \in \mathcal{A}$ and $cf \in \mathcal{A}$ if $c \in \mathbb{R}$.

2. $\mathcal{A}$ separates points of $[0,1]$ since the function $f(x) = e^x \in \mathcal{A}$ which is strictly monotone so that $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2$.

3. $\mathcal{A}$ vanishes at no point of $[0,1]$ since the function $f(x) = e^x \in \mathcal{A}$ which is non-zero at every point of $[0,1]$.

By the Stone Theorem, $\mathcal{A}$ is dense in $\mathscr{C}([0,1];\mathbb{R})$. □