Exercise Problem Sets 5

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Problem 1. Complete the following.

1. Suppose that $f_k, f, g : [0,\infty) \to \mathbb{R}$ are functions such that

   (a) $\forall\, R > 0$, f_k and g are Riemann integrable on $[0, R]$;

   (b) $|f_k(x)| \leqslant g(x)$ for all $k \in \mathbb{N}$ and $x \in [0,\infty)$;

   (c) $\forall\, R > 0$, $\{f_k\}_{k=1}^\infty$ converges to f uniformly on $[0, R]$;

   (d) $\displaystyle\int_0^\infty g(x)dx \equiv \lim_{R\to\infty} \int_0^R g(x)dx < \infty$.

   Show that $\displaystyle\lim_{k\to\infty} \int_0^\infty f_k(x)dx = \int_0^\infty f(x)dx$; that is,
   $$\lim_{k\to\infty} \lim_{R\to\infty} \int_0^R f_k(x)dx = \lim_{R\to\infty} \lim_{k\to\infty} \int_0^R f_k(x)dx\,.$$

2. Let $f_k(x)$ be given by $f_k(x) = \begin{cases} 1 & \text{if } k-1 \leqslant x < k\,, \\ 0 & \text{otherwise.} \end{cases}$ Find the (pointwise) limit f of the sequence $\{f_k\}_{k=1}^\infty$, and check whether $\displaystyle\lim_{k\to\infty} \int_0^\infty f_k(x)dx = \int_0^\infty f(x)dx$ or not. Briefly explain why one can or cannot apply 1.

3. Let $f_k : [0,\infty) \to \mathbb{R}$ be given by $f_k(x) = \dfrac{x}{1+kx^4}$. Find $\displaystyle\lim_{k\to\infty} \int_0^\infty f_k(x)dx$.

Proof. 1. First we note that since $|f_k(x)| \leqslant g(x)$ for all $x \in \mathbb{R}$, passing to the limit as $k \to \infty$ shows that $|f(x)| \leqslant g(x)$ for all $x \in \mathbb{R}$.

Let $\varepsilon > 0$ be given. Since $\displaystyle\lim_{R\to\infty} \int_0^R g(x)\,dx = \int_0^\infty g(x)\,dx$ exists, there exists $M > 0$ such that
$$\int_R^\infty g(x)\,dx = \Big| \int_0^R g(x)\,dx - \int_0^\infty g(x)\,dx \Big| < \frac{\varepsilon}{3} \qquad \forall\, R \geqslant M\,.$$

Since $\{f_k\}_{k=1}^\infty$ converges uniformly on $[0, M]$, $\displaystyle\lim_{k\to\infty} \int_0^M f_k(x)\,dx = \int_0^M f(x)\,dx$; thus there exists $N \geqslant 0$ such that
$$\Big| \int_0^M f_k(x)\,dx - \int_0^M f(x)\,dx \Big| < \frac{\varepsilon}{3} \quad \text{whenever} \quad k \geqslant N\,.$$

Therefore, if $k \geqslant N$, we have
$$\begin{aligned} &\Big| \int_0^\infty f_k(x)\,dx - \int_0^\infty f(x)\,dx \Big| \\ &\quad \leqslant \Big| \int_0^M f_k(x)\,dx - \int_0^M f(x)\,dx \Big| + \int_M^\infty |f(x)|\,dx + \int_M^\infty |f_k(x)|\,dx \\ &\quad < \frac{\varepsilon}{3} + 2\int_M^\infty g(x)\,dx < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon\,. \end{aligned}$$

thus $\lim_{k\to\infty} \int_0^\infty f_k(x)\,dx = \int_0^\infty f(x)\,dx$. This implies that

$$\lim_{k\to\infty}\lim_{R\to\infty}\int_0^R f_k(x)dx = \lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = \int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx$$
$$= \lim_{R\to\infty}\lim_{k\to\infty}\int_0^R f_k(x)dx\,.$$

2. If $x \in [0,\infty)$, we have $x \leqslant N$ for some $N \in \mathbb{N}$ (by the Archimedean property); thus for $k \geqslant N$ we have $f_k(x) = 0$. In other words, $\{f_k\}_{k=1}^\infty$ converges pointwise to the zero function. Let f be the zero function. Then
$$\int_0^\infty f_k(x)\,dx = \int_{k-1}^k 1\,dx = 1$$
so that $\lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = 1 \neq 0 = \int_0^\infty f(x)\,dx$. This is because we cannot find an integrable g satisfying that $|f_k(x)| \leqslant g(x)$ for all $x \in [0,\infty)$. In fact, if $|f_k(x)| \leqslant g(x)$ for all $x \in [0,\infty)$, then $g(x) \geqslant 1$ for all $x \in [0,\infty)$.

3. Let $g(x) = \dfrac{x}{1+x^4}$. Then $|f_k(x)| \leqslant g(x)$ for all $x \in [0,\infty)$ and $k \in \mathbb{N}$. Since $g(x) \leqslant x$ for $x \in [0,1]$ and $g(x) \leqslant \frac{1}{x^3}$ for $x \geqslant 1$, we find that
$$\int_0^\infty g(x)\,dx \leqslant \int_0^1 x\,dx + \int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{2} + \frac{1}{2} = 1 < \infty\,.$$
Moreover,
$$f_k'(x) = \frac{1+kx^4-4kx^4}{(1+kx^4)^2} = \frac{1-3kx^4}{(1+kx^4)^2}$$
which implies that for each $R > 0$,
$$\sup_{x\in[0,R]}|f_k(x)| \leqslant |f_k(0)| + |f_k(R)| + \left|\frac{(3k)^{-\frac{1}{4}}}{1+k\cdot\frac{1}{3k}}\right| = \frac{R}{1+kR^4} + \frac{3}{4}\Big(\frac{1}{3k}\Big)^{\frac{1}{4}}.$$
Therefore, the Sandwich Lemma implies that $\lim_{k\to\infty}\sup_{x\in[0,R]}|f_k(x)| = 0$ which shows that $\{f_k\}_{k=1}^\infty$ converges uniformly to the zero function on $[0,R]$ for every $R > 0$. By 1,
$$\lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = 0\,.$$
□

Problem 2. A series is called a ***power series about*** c or ***centered at*** c if it is of the form $\sum_{k=0}^\infty a_k(x-c)^k$ for some sequence $\{a_k\}_{k=0}^\infty \subseteq \mathbb{R}$ (or $\mathbb{C}$) and $c \in \mathbb{R}$ (or $\mathbb{C}$).

1. Show that if a power series centered at c is convergent at some point $b \neq c$, then the power series converges pointwise on $B(c,|b-c|)$.

2. Suppose that the power series $\sum_{k=0}^\infty a_k(x-c)^k$ converges pointwise in $B(c,R)$ for some $R > 0$.

(a) Suppose that $K \subseteq B(c, R)$ is a compact set. Show that $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on K.

(b) Show that $\sum\limits_{k=0}^{\infty} (k+1)a_{k+1}(x-c)^k$ converges pointwise on $B(c, R)$.

(c) Show that $\sum\limits_{k=1}^{\infty} \frac{a_{k-1}}{k}(x-c)^k$ converges pointwise on $B(c, R)$.

Proof. 1. Since the series $\sum\limits_{k=0}^{\infty} a_k(b-c)^k$ converges, $|a_k||b-c|^k \to 0$ as $k \to \infty$; thus there exists $M > 0$ such that $|a_k||b-c|^k \leqslant M$ for all k. $x \in B(c, |b-c|)$, the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely since

$$\sum_{k=0}^{\infty} |a_k(x-c)^k| \leqslant \sum_{k=0}^{\infty} |a_k||x-c|^k = \sum_{k=0}^{\infty} |a_k||b-c|^k \frac{|x-c|^k}{|b-c|^k} \leqslant M \sum_{k=0}^{\infty} \Big(\frac{|x-c|}{|b-c|}\Big)^k$$

which converges (because of the geometric series test or ratio test).

2. (a) Let $K \subseteq B(c, |b-c|)$ be a compact set. Then

$$\text{dist}(K, \partial B(c, |b-c|)) \equiv \inf\big\{|x-y| \,\big|\, x \in K, |y-c| = |b-c|\big\} > 0\,.$$

Define $r = \dfrac{|b-c| - \text{dist}(K, \partial B(c, |b-c|))}{|b-c|}$. Then $0 \leqslant r < 1$, and $|x-c| \leqslant r|b-c|$ for all $x \in K$. Therefore, $|a_k(x-c)^k| \leqslant Mr^k$ if $x \in K$; thus the Weierstrass M-test implies that the series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on K.

(b) By (a), it suffices to show that the power series $\sum\limits_{k=0}^{\infty} (k+1)a_{k+1}(x-c)^k$ converges pointwise on $B(c, R)$. Clearly the series converges at $x = c$. Let $x \in B(c, R)$ and $x \neq c$. Since $|x-c| < R$, there exists $b \in B(c, R)$ such that

$$|b-c| = \frac{R + |x-c|}{2}\,.$$

Then if $r = \dfrac{|x-c|}{|b-c|}$, $0 < r < 1$ and

$$\sum_{k=0}^{\infty} (k+1)|a_{k+1}||x-c|^k \leqslant \sum_{k=0}^{\infty} (k+1)|a_{k+1}||b-c|^k \Big(\frac{|x-c|}{|b-c|}\Big)^k \leqslant M \sum_{k=0}^{\infty} (k+1)r^k$$

for some $M > 0$. Note that the ratio test implies that the series $\sum\limits_{k=0}^{\infty} (k+1)r^k$ converges if $0 < r < 1$; thus $\sum\limits_{k=0}^{\infty} (k+1)|a_{k+1}||x-c|^k$ converges by the comparison test.

(c) By the same setting of (b), if $r = \dfrac{|x-c|}{|b-c|}$, $0 < r < 1$ and

$$\sum_{k=1}^{\infty} \frac{|a_{k-1}|}{k}|x-c|^k \leqslant \sum_{k=0}^{\infty} \frac{|a_{k-1}|}{k}|b-c|^k \Big(\frac{|x-c|}{|b-c|}\Big)^k \leqslant M \sum_{k=0}^{\infty} \frac{1}{k} r^k$$

for some $M > 0$. Note that the ratio test implies that the series $\sum\limits_{k=0}^{\infty} \frac{1}{k} r^k$ converges if $0 < r < 1$; thus $\sum\limits_{k=0}^{\infty} \frac{a_{k-1}}{k}|x-c|^k$ converges by the comparison test. $\square$

Remark: From the problem above, we have the following:

1. Part 1 implies that the interior of the collection of all x at which the power series converges is either an open ball or empty. The radius of such a ball is called the **radius of convergence**.
2. Part 2(a) and 2(b) imply that $\frac{d}{dx}\sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=0}^{\infty} \frac{d}{dx} a_k(x-c)^k$ for all $x \in B(c,R)$ if $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges pointwise in $B(c,R)$.
3. Part 2(a), 2(b) and 2(c) imply that $\frac{d}{dx}\sum_{k=1}^{\infty} \frac{a_{k-1}}{k}(x-c)^k = \sum_{k=0}^{\infty} a_k(x-c)^k$ for all $x \in B(c,R)$ if $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges pointwise in $B(c,R)$.

Problem 3. In this problem we investigate the differentiability of a power series in a different way.

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$, and $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a power series with radius of convergence $R > 0$. Let $s_n(x) = \sum_{k=0}^{n} a_k x^k$ be the n-th partial sum, $R_n(x) = f(x) - s_n(x)$, and $g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$. For $x, x_0 \in B[0,\rho] \subsetneq B(0,R)$, where $x \neq x_0$, write

$$\frac{f(x)-f(x_0)}{x-x_0} - g(x) = \frac{s_n(x)-s_n(x_0)}{x-x_0} - s_n'(x_0) + \big(s_n'(x_0) - g(x_0)\big) + \frac{R_n(x)-R_n(x_0)}{x-x_0}\,. \tag{0.1}$$

Show that

$$\Big|\frac{R_n(x)-R_n(x_0)}{x-x_0}\Big| \leqslant \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}\,,$$

and use the inequality above to show that $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = g(x_0)$.

Proof. Let R be the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$.

Claim: The series $\sum_{k=1}^{\infty} k|a_k|\rho^{k-1}$ converges for all $0 < \rho < R$.

To see the claim, we note that for each $0 < r < R$, $\sum_{k=0}^{\infty} a_k r^k$ converges; thus $\lim_{k\to\infty} a_k r^k = 0$. This implies that the sequence $\{a_k r^k\}_{k=1}^{\infty}$ is bounded for all $0 < r < R$. Let $M(r)$ denote a real number satisfying $|a_k r^k| \leqslant M(r)$ for all $k \in \mathbb{N} \cup \{0\}$. Then for $0 < \rho < R$, we choose r so that $0 < \rho < r < R$ so that

$$\sum_{k=1}^{\infty} k|a_k|\rho^{k-1} = \sum_{k=1}^{\infty} k|a_k|r^{k-1}\Big(\frac{\rho}{r}\Big)^{k-1} \leqslant M(r)\sum_{k=1}^{\infty} k\Big(\frac{\rho}{r}\Big)^{k-1}$$

where the convergence of the series on the right-hand side can be obtained by the ratio test. The claim is then established by the comparison test.

Since $R_n(x) = \sum\limits_{k=n+1}^{\infty} a_k x^k$ converges for all $x \in (-R, R)$, for $x \neq x_0$ we have

$$\begin{aligned}\frac{R_n(x) - R_n(x_0)}{x - x_0} &= \frac{1}{x - x_0} \sum_{k=n+1}^{\infty} a_k (x^k - x_0^k) \\ &= \sum_{k=n+1}^{\infty} a_k \big(x^{k-1} + x^{k-2} x_0 + \cdots + x x_0^{k-2} + x_0^{k-1} \big) \,;\end{aligned}$$

thus if $x, x_0 \in [-\rho, \rho] \subseteq (-R, R)$ and $x \neq x_0$,

$$\begin{aligned}\Big| \frac{R_n(x) - R_n(x_0)}{x - x_0} \Big| &\leqslant \sum_{k=n+1}^{\infty} |a_k| \big(|x|^{k-1} + |x|^{k-2}|x_0| + \cdots + |x||x_0|^{k-2} + |x_0|^{k-1} \big) \\ &\leqslant \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1} \,.\end{aligned}$$

Let $\varepsilon > 0$ be given. By the claim above there exists $N > 0$ such that if $n \geqslant N$,

$$\sum_{k=n+1}^{\infty} k|a_k||x_0|^{k-1} < \frac{\varepsilon}{3} \qquad \text{and} \qquad \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1} < \frac{\varepsilon}{3} \,.$$

Therefore, (0.1) implies that

$$\begin{aligned}&\Big| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \Big| \\ &\quad \leqslant \Big| \frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0) \Big| + \big| s_n'(x_0) - g(x_0) \big| + \Big| \frac{R_n(x) - R_n(x_0)}{x - x_0} \Big| \\ &\quad \leqslant \Big| \frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0) \Big| + \Big| \sum_{k=n+1}^{\infty} k a_k x_0^{k-1} \Big| + \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1} \\ &\quad \leqslant \Big| \frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0) \Big| + \frac{2\varepsilon}{3} \,;\end{aligned}$$

thus

$$\begin{aligned}\limsup_{x \to x_0} \Big| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \Big| &\leqslant \limsup_{x \to x_0} \Big| \frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0) \Big| + \frac{2\varepsilon}{3} \\ &= \lim_{x \to x_0} \Big| \frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0) \Big| + \frac{2\varepsilon}{3} < \varepsilon \,.\end{aligned}$$

Since $\varepsilon > 0$ is given arbitrarily, we find that $\lim\limits_{x \to x_0} \Big| \dfrac{f(x) - f(x_0)}{x - x_0} - g(x_0) \Big| = 0$ which shows that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g(x_0) \,. \qquad \square$$

Problem 4. Suppose that the series $\sum\limits_{n=0}^{\infty} a_n = 0$, and $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ for $-1 < x \leqslant 1$. Show that f is continuous at $x = 1$ by complete the following.

1. Write $s_n = a_0 + a_1 + \cdots + a_n$ and $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$. Show that

$$S_n(x) = (1-x)(s_0 + s_1 x + \cdots + s_{n-1}x^{n-1}) + s_n x^n$$

and $f(x) = (1-x)\sum\limits_{n=0}^{\infty} s_n x^n$.

2. Using the representation of f from above to conclude that $\lim\limits_{x\to 1^-} f(x) = 0$.

3. What if $\sum\limits_{n=0}^{\infty} a_n$ is convergent but not zero?

Proof. 1. Let $s_n = a_0 + a_1 + \cdots + a_n$ and $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

$$\begin{aligned} S_n(x) &= \sum_{k=0}^{n} a_k x^k = a_0 + \sum_{k=1}^{n} a_k x^k = s_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k \\ &= s_0 + \sum_{k=1}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k = \sum_{k=0}^{n} s_k x^k - \sum_{k=0}^{n-1} s_k x^{k+1} \\ &= s_n x^n + \sum_{k=0}^{n-1} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k \\ &= (1-x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n \,. \end{aligned}$$

Therefore, by the fact that $\lim\limits_{n\to\infty} s_n = 0$, we find that if $x \in (-1, 1]$,

$$f(x) = \lim_{n\to\infty} S_n(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k \,.$$

2. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} s_n = 0$, there exists $N > 0$ such that $|s_n| < \dfrac{\varepsilon}{2}$ for all $n \geqslant N$. Choose $0 < \delta < 1$ such that $\delta \sum\limits_{k=0}^{N-1} |s_k| < \dfrac{\varepsilon}{2}$. Then if $1 - \delta < x < 1$,

$$\begin{aligned} \big|f(x)\big| &\leqslant |1-x| \sum_{k=0}^{N-1} |s_k||x|^k + |1-x| \sum_{k=N}^{\infty} |s_k||x|^k \\ &\leqslant \delta \sum_{k=0}^{N-1} |s_k| + \frac{\varepsilon}{2}|1-x||x|^N \sum_{k=0}^{\infty} |x|^k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}|1-x| \frac{1}{1-|x|} = \varepsilon \,. \end{aligned}$$

Therefore, $\lim\limits_{x\to 1^-} f(x) = 0 = f(1)$ which shows that f is continuous at 1.

3. If $s = \sum\limits_{k=0}^{\infty} a_k \neq 0$, we define a new series $\sum\limits_{n=0}^{\infty} b_n x^n$ by $b_0 = a_0 - s$ and $b_n = a_n$ for all $n \in \mathbb{N}$. Then $g(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ also converges for $x \in (-1, 1]$ and satisfies that $g(1) = 0$. Therefore, 1 and 2 imply that g is continuous at 1; thus $\lim\limits_{x\to 1^-} g(x) = 0$. By the fact that $g(x) = f(x) - s$, we conclude that

$$\lim_{x\to 1^-} f(x) = s = \sum_{n=0}^{\infty} a_n = f(1) \,.$$

□

Problem 5. Show that the series
$$\sum_{k=1}^{\infty}(-1)^k\frac{x^2+k}{k^2}$$
converges uniformly on every bounded interval.

Proof. Since $\sum\limits_{k=1}^{\infty}(-1)^k\frac{1}{k} = -\ln 2$ converges (by the Dirichlet test), we have
$$\sum_{k=1}^{\infty}(-1)^k\frac{x^2+k}{k^2} = \sum_{k=1}^{\infty}(-1)^k\frac{x^2}{k^2} - \ln 2 \qquad \forall\, x \in \mathbb{R}\,.$$

Let $M_k = \dfrac{R^2}{k^2}$. Then

1. $\sup\limits_{x\in[-R,R]}\left|(-1)^k\frac{x^2}{k^2}\right| \leqslant M_k$ for all $k \in \mathbb{N}$.
2. $\sum\limits_{k=1}^{\infty} M_k < \infty$ (by the integral test).

Therefore, the Weierstrass M-test implies that $\sum\limits_{k=1}^{\infty}(-1)^k\frac{x^2}{k^2}$ converges uniformly on $[-R,R]$. □

Problem 6. Consider the function
$$f(x) = \sum_{k=1}^{\infty}\frac{1}{1+k^2x}\,.$$
On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? If f bounded?

Problem 7. Determine which of the following real series $\sum\limits_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x) = \begin{cases} 0 & \text{if } x \leqslant k\,, \\ (-1)^k & \text{if } x > k\,. \end{cases}$
2. $g_k(x) = \begin{cases} \dfrac{1}{k^2} & \text{if } |x| \leqslant k\,, \\ \dfrac{1}{x^2} & \text{if } |x| > k\,. \end{cases}$
3. $g_k(x) = \dfrac{(-1)^k}{\sqrt{k}}\cos(kx)$ on $\mathbb{R}$.

Proof. 1. By the definition of g_k, we find that the partial sum $S_n(x) = \sum\limits_{k=1}^{n} g_k(x)$ satisfies that for all $n \in \mathbb{N}$,
$$S_{2n}(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-1,2n]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$
and
$$S_{2n-1}(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-3,2n-2] \cup (2n-1,\infty)\,, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ converges pointwise to the function

$$S(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-3, 2n-2] \cup \cdot\,, \\ 0 & \text{otherwise} \end{cases}$$

or more precisely,

$$S(x) = \sum_{k=1}^{\infty} \mathbf{1}_{(2k-1,2k]}(x)\,.$$

The convergence is uniformly on any bounded subset of $\mathbb{R}$, and the limit function S has discontinuities on $\mathbb{N}$.

2. Let $M_k = \dfrac{1}{k^2}$. Then $\sup\limits_{x\in\mathbb{R}} |g_k(x)| \leqslant M_k$ and $\sum\limits_{k=1}^{\infty} M_k$ converges (by the integral test). Therefore, the Weierstrass M-test implies that $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on $\mathbb{R}$.

3. If $x = (2n+1)\pi$ for some $n \in \mathbb{Z}$, then $\cos(kx) = (-1)^k$ for all $k \in \mathbb{N}$; thus $\sum\limits_{k=1}^{\infty} g_k(x)$ diverges at $x = (2n+1)\pi$ (by the integral test).

Now suppose that $x \notin \{(2n+1)\pi \,|\, n \in \mathbb{Z}\}$. Let $S_n(x) = \sum\limits_{k=1}^{n} (-1)^k \cos(kx)$. Then $S_n(x) = \sum\limits_{k=1}^{n} \cos(k(x+\pi))$ and

$$\begin{aligned} 2\sin\frac{x+\pi}{2} S_n(x) &= \sum_{k=1}^{n} \Big[\sin\big(k+\frac{1}{2}\big)(x+\pi) - \sin\big(k-\frac{1}{2}\big)(x+\pi)\Big] \\ &= \sin\big(n+\frac{1}{2}\big)(x+\pi) - \sin\frac{x+\pi}{2}\,; \end{aligned}$$

thus

$$S_n(x) = \frac{(-1)^n \cos(n+\frac{1}{2})x}{2\cos\frac{x}{2}} - \frac{1}{2} \qquad \forall\, x \in \mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\,.$$

The equality above shows that

$$|S_n(x)| \leqslant \frac{1}{2|\cos\frac{x}{2}|} + \frac{1}{2} \qquad \forall\, x \in \mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\,,$$

which is bounded independent of n. The Dirichlet test then shows that $\sum\limits_{k=1}^{\infty} g_k(x)$ converges for all $x \in \mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}$. Therefore, $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise on $\mathbb{R}\backslash\{(2n+1)\pi \,|\, n \in \mathbb{Z}\}$.

Let $A \subseteq \mathbb{R}$ be a set satisfying that

$$d\big(x, \{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\big) = \inf\big\{|x-y| \,\big|\, y \in \{(2n+1)\pi \,|\, n \in \mathbb{Z}\}\big\} \geqslant \delta \qquad \forall\, x \in A\,.$$

Then the computation above shows that $|S_n(x)| \leqslant R \equiv \dfrac{1}{2|\cos\frac{\delta}{2}|} + \dfrac{1}{2}$ for all $x \in A$. If $n > m$, we have

$$\begin{aligned}
\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx) &= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} \big[S_k(x) - S_{k-1}(x)\big] \\
&= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_k(x) - \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_{k-1}(x) \\
&= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_k(x) - \sum_{k=m}^{n-1} \frac{1}{\sqrt{k+1}} S_k(x) \\
&= \frac{1}{\sqrt{n}} S_n(x) - \frac{1}{\sqrt{m+1}} S_m(x) + \sum_{k=m+1}^{n-1} \Big(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\Big) S_k(x)\,;
\end{aligned}$$

thus if $x \in A$,

$$\Big|\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx)\Big| \leqslant \Big[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m+1}} + \sum_{k=m+1}^{n-1} \Big(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\Big)\Big] R = \frac{2R}{\sqrt{m+1}}\,.$$

Therefore, for a given $\varepsilon > 0$, by choosing $N > 0$ satisfying $\dfrac{2R}{\sqrt{N+1}} < \varepsilon$ we conclude that

$$\Big|\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx)\Big| < \varepsilon \quad \text{whenever } n > m \geqslant N \text{ and } x \in A\,.$$

By the Cauchy criterion, $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on A; thus $\sum\limits_{k=1}^{\infty} g_k$ is continuous at every point at which the series converges. □