Exercise Problem Sets 4

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Problem 1. Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be Riemann measurable sets, and $f : A \times B \to \mathbb{R}$ be non-negative, uniformly continuous and integrable on $A \times B$. Define $F(x) = \displaystyle\int_B f(x, y)\, dy$.

1. Show that if B is bounded, then $F : A \to \mathbb{R}$ is continuous. How about if B is not bounded?

2. Let f have the additional property that for each $\varepsilon > 0$, there exists $N > 0$ such that
$$\Big| \int_{B \cap B(0,k)} (f \wedge k)(x, y)\, dy - \int_B f(x, y)\, dy \Big| < \varepsilon \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$
Show that F is continuous on A. In particular, show that if $f(x, y) \leqslant g(y)$ for all $(x, y) \in A \times B$, and g is integrable on B, then F is continuous.

Proof. 1. If B is bounded, then B has volume. Let $\varepsilon > 0$ be given. By the uniform continuity of f, there exists $\delta > 0$ such that

$$\big|f(x_1, y_1) - f(x_2, y_2)\big| < \frac{\varepsilon}{\nu(B) + 1} \quad \forall\, \big|(x_1, y_1) - (x_2, y_2)\big| < \delta \text{ and } x_1, x_2 \in A, y_1, y_2 \in B\,.$$

Therefore, if $|x_1 - x_2| < \delta$ and $x_1, x_2 \in A$,

$$\begin{aligned}\big|F(x_1) - F(x_2)\big| &= \Big| \int_B \big[f(x_1, y) - f(x_2, y)\big]\, dy \Big| \leqslant \int_B \big|f(x_1, y) - f(x_2, y)\big|\, dy \\ &\leqslant \int_B \frac{\varepsilon}{\nu(B) + 1}\, dx \leqslant \frac{\varepsilon \nu(B)}{\nu(B) + 1} < \varepsilon\,.\end{aligned}$$

This implies that F is uniformly continuous on A.

If B is unbounded, then the argument above does not apply. In fact, consider the case

$$f(x, y) = \frac{\sqrt{x}}{1 + x^2 y^2}\,, \quad A = [0, 1] \quad \text{and} \quad B = \mathbb{R}\,.$$

Then f is non-negative and uniformly continuous on $A \times B$ (by Problem 3 of Exercise 12 in the first semester). Note that $F(0) = 0$ while if $x > 0$,

$$F(x) = \int_{\mathbb{R}} f(x, y)\, dy = \int_{-\infty}^{\infty} \frac{\sqrt{x}}{1 + x^2 y^2}\, dy = \frac{\sqrt{x}}{x} \arctan(xy)\Big|_{y=-\infty}^{y=\infty} = \frac{\pi}{\sqrt{x}}\,.$$

Therefore, the Tonelli Theorem implies that

$$\int_{A \times B} f(x, y)\, d(x, y) = \int_A \Big(\int_B f(x, y)\, dy \Big)\, dx = \int_0^1 \frac{\pi}{\sqrt{x}}\, dx = 2\pi < \infty$$

which shows that f is integrable on $A \times B$. However, F is not continuous at $x = 0$.

2. Let $\varepsilon > 0$ be given. Since f has the property mentioned above, there exists $N > 0$ such that

$$\left|\int_{B\cap B(0,k)} (f\wedge k)(x,y)\,dy - \int_B f(x,y)\,dy\right| < \frac{\varepsilon}{3} \qquad \forall\, k \geqslant N \text{ and } x \in A\,.$$

By the uniform continuity of f on $A \times B$, there exists $\delta > 0$ such that

$$\big|f(x_1,y_1) - f(x_2,y_2)\big| < \frac{\varepsilon}{3} \qquad \forall\,\big|(x_1,y_1)-(x_2,y_2)\big| < \delta \text{ and } x_1,x_2 \in A, y_1,y_2 \in B\,.$$

Suppose that $|x_1 - x_2| < \delta$, $x_1, x_2 \in A$ and $y \in B$.

(a) If $f(x_1,y)$ and $f(x_2,y)$ are both not greater than N, then

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| = \big|f(x_1,y)-f(x_2,y)\big| < \varepsilon\,.$$

(b) If $f(x_1,y)$ and $f(x_2,y)$ are both greater than N, then

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| = |N-N| = 0\,.$$

(c) If one and only one of $f(x_1,y)$ and $f(x_2,y)$ is greater than N, then

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| < \big|f(x_1,y)-f(x_2,y)\big| < \varepsilon\,.$$

Case (a), (b) and (c) show that

$$\big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big| < \frac{\varepsilon}{3\nu(B(0,N))} \quad \forall\,|x_1-x_2|<\delta, x_1,x_2 \in A \text{ and } y \in B\,.$$

Therefore, if $x_1, x_2 \in A$ and $|x_1 - x_2| < \delta$,

$$\begin{aligned}
\big|F(x_1)-F(x_2)\big| &\leqslant \left|\int_{B\cap B(0,N)} (f\wedge N)(x_1,y)\,dy - \int_B f(x_1,y)\,dy\right| \\
&\quad + \left|\int_{B\cap B(0,N)} (f\wedge N)(x_2,y)\,dy - \int_B f(x_2,y)\,dy\right| \\
&\quad + \left|\int_{B\cap B(0,N)} (f\wedge N)(x_1,y)\,dy - \int_{B\cap B(0,N)} (f\wedge N)(x_2,y)\,dy\right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int_{B\cap B(0,N)} \big|(f\wedge N)(x_1,y) - (f\wedge N)(x_2,y)\big|\,dy \leqslant \varepsilon\,.
\end{aligned}$$

This implies that F is uniformly continuous on A.

Now suppose that $f(x,y) \leqslant g(y)$ for all $(x,y) \in A \times B$, and g is integrable on B. Then

$$\lim_{k\to\infty} \int_{B\cap B(0,k)} (g\wedge k)(y)\,dy = \int_B g(y)\,dy\,;$$

thus there exists $N > 0$ such that

$$\left|\int_{B\cap B(0,k)} (g\wedge k)(y)\,dy - \int_B g(y)\,dy\right| < \varepsilon \qquad \text{whenever} \quad k \geqslant N\,.$$

Therefore, for all $k \geqslant N$ and $x \in A$,

$$
\begin{aligned}
&\Big| \int_{B \cap B(0,k)} (f \wedge k)(x,y)\, dy - \int_B f(x,y)\, dy \Big| \\
&\quad \leqslant \Big| \int_{B \cap B(0,k)} (f \wedge k)(x,y)\, dy - \int_{B \cap B(0,k)} f(x,y)\, dy \Big| + \int_{B \cap B(0,k)^{\complement}} f(x,y)\, dy \\
&\quad \leqslant \int_{B \cap B(0,k)} \big|(f \wedge k)(x,y) - f(x,y)\big|\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&\quad \leqslant \int_{\{y \in B \cap B(0,k) \,|\, f(x,y) > k\}} \big[f(x,y) - k\big]\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&\quad \leqslant \int_{\{y \in B \cap B(0,k) \,|\, g(y) > k\}} \big[g(y) - k\big]\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&\quad \leqslant \int_{B \cap B(0,k)} \big[g(y) - (g \wedge k)(y)\big]\, dy + \int_{B \cap B(0,k)^{\complement}} g(y)\, dy \\
&\quad = \int_B g(y)\, dy - \int_{B \cap B(0,k)} (g \wedge k)(y)\, dy < \varepsilon \,.
\end{aligned}
$$

This shows that f satisfies the condition mentioned in 2, so F is continuous on A. ▫

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Riemann measurable function, and $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \int_{\mathbb{R}} f(y) \cos(x - y)\, dy$$

whenever the integral exists. Show that if the function f is integrable, then F is defined on $\mathbb{R}$ and is differentiable on $\mathbb{R}$ with derivative

$$F'(x) = \int_{\mathbb{R}} f(y) \frac{\partial}{\partial x} \cos(x - y)\, dy = -\int_{\mathbb{R}} f(y) \sin(x - y)\, dy \,.$$

Proof. Let $x \in \mathbb{R}$ be given. Since f is Riemann measurable, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(y) = f(y)\cos(x - y)$ is Riemann measurable and $|g(y)| \leqslant |f(y)|$ for all $y \in \mathbb{R}$. Since f is integrable, the comparison test implies that g is integrable. Therefore, F is defined everywhere on $\mathbb{R}$.

Let $\{h_k\}_{k=1}^{\infty}$ be a non-zero sequence with limit 0. Define

$$g_k(y) = f(y) \frac{\cos(x + h_k - y) - \cos(x - y)}{h_k} \,.$$

Then for all $y \in \mathbb{R}$, $\lim\limits_{k \to \infty} g_k(y) = f(y) \dfrac{\partial}{\partial x}(cos(x - y)) = -f(y) \sin(x - y)$.

Since $\Big|\dfrac{d}{dy} \cos x\Big| \leqslant 1$, the Mean Value Theorem implies that

$$\big|\cos(x + h_k - y) - \cos(x - y)\big| \leqslant |h_k| \,.$$

Therefore,

$$\big|g_k(y)\big| \leqslant \big|f(y)\big| \qquad \forall\, x \in \mathbb{R} \,.$$

Since f is integrable on $\mathbb{R}$, $|f|$ is integrable on $\mathbb{R}$; thus the Dominated Convergence Theorem implies that

$$\lim_{k \to \infty} \frac{F(x + h_k) - F(x)}{h_k} = \lim_{k \to \infty} \int_{\mathbb{R}} g_k(y)\, dy = -\int_{\mathbb{R}} f(x) \sin(x - y)\, dy \,.$$

The equality above shows that for each non-zero sequence $\{h_k\}_{k=1}^{\infty}$ with limit 0, the limit

$$\lim_{k\to\infty} \frac{F(x+h_k)-F(x)}{h_k} = -\int_{\mathbb{R}} f(x)\sin(x-y)\,dy$$

exists. By the definition of the limit of functions,

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = -\int_{\mathbb{R}} f(x)\sin(x-y)\,dy\,. \qquad \square$$

Problem 3. Let $f:\mathbb{R}\to\mathbb{R}$ be an integrable Riemann measurable function, and $F:\mathbb{R}\to\mathbb{R}$ be defined by

$$F(x) = \int_{\mathbb{R}} f(y)\cos(xy)\,dy$$

(which exists for all $x\in\mathbb{R}$ since f is integrable). Show that if the function $g(x)=xf(x)$ is integrable, then F is differentiable on $\mathbb{R}$ and

$$F'(y) = \int_{\mathbb{R}} f(x)\frac{\partial}{\partial y}\cos(xy)\,dx = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx\,.$$

Proof. Let $y\in\mathbb{R}$ be given, and $\{h_k\}_{k=1}^{\infty}$ be a non-zero sequence with limit 0. Define

$$g_k(x) = f(x)\frac{\cos(x(y+h_k))-\cos(xy)}{h_k}\,.$$

Then for all $x\in\mathbb{R}$, $\lim\limits_{k\to\infty} g_k(x) = f(x)\frac{\partial}{\partial y}(cos(xy)) = -xf(x)\sin(xy)$.

Since $\left|\frac{d}{dx}\cos x\right| \leqslant 1$, the Mean Value Theorem implies that

$$\big|\cos(x(y+h_k))-\cos(xy)\big| \leqslant |xh_k|\,.$$

Therefore,

$$\big|g_k(x)\big| \leqslant \big|xf(x)\big| = \big|g(x)\big| \qquad \forall\, x\in\mathbb{R}\,.$$

Since g is integrable on $\mathbb{R}$, $|g|$ is integrable on $\mathbb{R}$; thus the Dominated Convergence Theorem implies that

$$\lim_{k\to\infty}\frac{F(y+h_k)-F(y)}{h_k} = \lim_{k\to\infty}\int_{\mathbb{R}} h_k(x)\,dx = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx\,.$$

The equality above shows that for each non-zero sequence $\{h_k\}_{k=1}^{\infty}$ with limit 0, the limit

$$\lim_{k\to\infty}\frac{F(y+h_k)-F(y)}{h_k} = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx$$

exists. By the definition of the limit of functions,

$$\lim_{h\to 0}\frac{F(y+h)-F(y)}{h} = -\int_{\mathbb{R}} xf(x)\sin(xy)\,dx\,. \qquad \square$$

Problem 4. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{e^{-xy}\sin y}{y} & \text{if } y\neq 0\,,\\ 1 & \text{if } y=0\,.\end{cases}$$

Complete the following.

1. Show that $f_x(x,y)$ is continuous everywhere, and show that $f(x,\cdot)$ is integrable on $[0,\infty)$ for all $x>0$.

2. Define $F(x)=\displaystyle\int_0^\infty f(x,y)\,dy$ for $x>0$. Show that $F'(x)=-\dfrac{1}{x^2+1}$.

3. Show that $F(x)=\dfrac{\pi}{2}-\tan^{-1}x$ if $x>0$, and conclude that

$$\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}\,.$$

Proof. 1. Note that if $y\neq 0$, $f_x(x,y)=e^{-xy}\sin y$ while $f_x(x,0)=0$. Clearly f_x is continuous on $\mathbb{R}^2$ except perhaps on the x-axis. On the other hand, since $\displaystyle\lim_{(x,y)\to(a,0)} f(x,y)=0$, we conclude that f_x is also continuous on the x-axis. Therefore, f_x is continuous everywhere.

Let $x>0$ be given. Then $\big|f(x,y)\big|\leqslant e^{-xy}$. Since the right-hand side function, for given $x>0$, is integrable on $[0,\infty)$, the comparison test implies that $f(x,\cdot)$ is integrable on $[0,\infty)$.

2. Let $x>0$ be given, and $\{h_k\}_{k=1}^\infty$ be a non-zero sequence with limit 0. W.L.O.G., we can assume that $|h_k|<\dfrac{x}{2}$ since $x>0$. Define

$$g_k(y)=\left\{\begin{array}{cl}\dfrac{e^{-yh_k}-1}{h_k}e^{-xy}\dfrac{\sin y}{y} & \text{if } y\neq 0\,,\\ 0 & \text{if } y=0\,.\end{array}\right.$$

The Mean Value Theorem implies that $\left|\dfrac{e^{-yh_k}-1}{h_k}\right|\leqslant e^{\frac{xy}{2}}|y|$; thus

$$\big|g_k(y)\big|\leqslant e^{-\frac{xy}{2}}\qquad \forall\, y\geqslant 0\,.$$

Since the right-hand side function, for given $x>0$, is integrable on $[0,\infty)$, the Dominated Convergence Theorem implies that

$$\begin{aligned}\lim_{k\to\infty}\frac{F(x+h_k)-F(x)}{h_k}&=\lim_{k\to\infty}\int_0^\infty \frac{f(x+h_k,y)-f(x,y)}{h_k}\,dy=\lim_{k\to\infty}\int_0^\infty g_k(y)\,dy\\ &=\int_0^\infty \lim_{k\to\infty} g_k(y)\,dy=-\int_0^\infty e^{-xy}\sin y\,dy\end{aligned}$$

Integrating by parts, by the fact $x>0$ we find that

$$\begin{aligned}\int_0^\infty e^{-xy}\sin y\,dy&=-e^{-xy}\cos y\Big|_{y=0}^{y=\infty}-x\int_0^\infty e^{-xy}\cos y\,dy\\ &=1-x\Big[e^{-xy}\sin y\Big|_{y=0}^{y=\infty}+x\int_0^\infty e^{-xy}\sin y\,dy\Big]\\ &=1-x^2\int_0^\infty e^{-xy}\sin y\,dy\,;\end{aligned}$$

thus we conclude that

$$\lim_{k\to\infty}\frac{F(x+h_k)-F(x)}{h_k}=-\frac{1}{1+x^2}$$

for all $x > 0$ and non-zero sequence $\{h_k\}_{k=1}^{\infty}$ with limit 0. Therefore, for $x > 0$ the limit $\lim\limits_{h\to 0} \dfrac{F(x+h)-F(x)}{h}$ exists (so that F is differentiable on $(0,\infty)$) and

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \frac{1}{1+x^2} \qquad \forall\, x > 0\,.$$

3. By the (generalized version of) Fundamental Theorem of Calculus, for $a, b > 0$ we have

$$F(b) - F(a) = \int_a^b F'(x)\,dx = -\int_a^b \frac{1}{1+x^2}\,dx = \arctan x\Big|_{x=a}^{x=b} = \arctan a - \arctan b\,.$$

Note that for $a > 0$ we have

$$|F(a)| \leqslant \int_0^\infty e^{-ay}\,dy = \frac{e^{-ay}}{-a}\Big|_{y=0}^{y=\infty} = \frac{1}{a}\,;$$

thus $\lim\limits_{a\to\infty} F(a) = 0$ by the Sandwich lemma. Therefore, for $x > 0$,

$$F(x) = \lim_{a\to\infty}\big[F(x) - F(a)\big] = \lim_{a\to\infty}\big(\arctan a - \arctan x\big) = \frac{\pi}{2} - \arctan x\,.$$

Finally, we show that $F(0) = \lim\limits_{x\to 0^+} F(x)$. Let $\varepsilon > 0$ be given. Since

$$\frac{\partial}{\partial y}\Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big) = (e^{-xy}-1)\sin y\,,$$

integrating by parts shows that for all $n > 0$,

$$\begin{aligned}\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy &= \frac{1}{y}\Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big)\Big|_{y=n}^{y=\infty} \\ &\quad + \int_n^\infty \Big(\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big)\frac{1}{y^2}\,dy\,.\end{aligned}$$

By the fact that

$$\Big|\frac{-e^{-xy}\cos y - xe^{-xy}\sin y}{x^2+1} + \cos y\Big| \leqslant \frac{x+1}{x^2+1} + 1 \leqslant \frac{5}{2} < 3\,,$$

we have

$$\Big|\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \leqslant \int_n^\infty \frac{3}{y^2}\,dy + \frac{3}{n} = \frac{6}{n}\,.$$

Therefore, for all $n > 0$,

$$\begin{aligned}\big|F(x) - F(0)\big| &= \Big|\int_0^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \\ &\leqslant \Big|\int_0^n (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| + \Big|\int_n^\infty (e^{-xy}-1)\frac{\sin y}{y}\,dy\Big| \\ &\leqslant \int_0^n (1-e^{-xy})\,dy + \frac{6}{n} = n + \frac{e^{-nx}-1}{x} + \frac{6}{n}\end{aligned}$$

so that

$$\limsup_{x\to 0^+}\big|F(x) - F(0)\big| \leqslant \frac{6}{n} \qquad \forall\, n > 0\,.$$

Since $n > 0$ is given arbitrarily, we conclude that $\limsup_{x \to 0^+} \big|F(x) - F(0)\big| = 0$ which shows that $\lim_{x \to 0^+} F(x) = F(0)$. As a consequence,

$$\int_0^\infty \frac{\sin x}{x}\, dx = F(0) = \lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \big(\frac{\pi}{2} - \arctan x\big) = \frac{\pi}{2}\,. \qquad \square$$

Problem 5. Let (M, d) and (N, ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be a sequence of functions such that for some function $f : A \to N$, we have that for all $x \in A$, if $\{x_k\}_{k=1}^\infty \subseteq A$ and $x_k \to x$ as $k \to \infty$, then

$$\lim_{k \to \infty} f_k(x_k) = f(x)\,.$$

Show that

1. $\{f_k\}_{k=1}^\infty$ converges pointwise to f.
2. If $\big\{f_{k_j}\big\}_{j=1}^\infty$ is a subsequence of $\{f_k\}_{k=1}^\infty$, and $\{x_j\}_{j=1}^\infty \subseteq A$ is a convergent sequence satisfying that $\lim_{j \to \infty} x_j = x$, then
$$\lim_{j \to \infty} f_{k_j}(x_j) = f(x)\,.$$
3. Show that if in addition A is compact and f is continuous on A, then $\{f_k\}_{k=1}^\infty$ converges uniformly f on A.

Proof. 1. Let $x \in A$ be given. Define $\{x_k\}_{k=1}^\infty$ by $x_k = x$ for all $k \in \mathbb{N}$. Then $\lim_{k \to \infty} x_k = x$; thus

$$\lim_{k \to \infty} f_k(x) = \lim_{k \to \infty} f_k(x_k) = f(x)$$

which shows that $\{f_k\}_{k=1}^\infty$ converges pointwise to f.

2. Let $\{f_{k_j}\}_{j=1}^\infty$ be a subsequence of $\{f_k\}_{k=1}^\infty$, and $\{x_j\}_{j=1}^\infty$ be a convergent sequence with limits x. Define a new sequence $\{y_\ell\}_{\ell=1}^\infty$ by

$$y_1, \cdots, y_{k_1} = x_1\,,\ y_{k_1+1}, \cdots, y_{k_2} = x_2\,,\ \cdots,\ y_{k_\ell+1}, \cdots, y_{k_{\ell+1}} = x_{\ell+1}\,, \cdots;$$

that is, the first k_1 terms of $\{y_\ell\}_{\ell=1}^\infty$ is x_1, the next $(k_2 - k_1)$ terms of $\{y_\ell\}_{\ell=1}^\infty$ is x_2, and so on. Then $\{y_\ell\}_{\ell=1}^\infty$ converges to x;

$$\lim_{\ell \to \infty} f_\ell(y_\ell) = f(x)\,.$$

Since $\big\{f_{k_j}(x_j)\big\}_{j=1}^\infty$ is a subsequence of $\big\{f_\ell(y_\ell)\big\}_{\ell=1}^\infty$, $\lim_{j \to \infty} f_{k_j}(x_j) = f(x)$.

3. Suppose the contrary that $\{f_k\}_{k=1}^\infty$ does not converge uniformly to f on A. Then there exists $\varepsilon > 0$ such that for each $k > 0$ there exist $n_k \geqslant k$ (W.L.O.G. we can assume that $n_{k+1} > n_k$ for all $k \in \mathbb{N}$) and $x_k \in A$ such that

$$\rho\big(f_{n_k}(x_k), f(x_k)\big) \geqslant \varepsilon\,.$$

By the compactness of A, there exists a convergent subsequence $\{x_{k_j}\}_{j=1}^\infty$ of $\{x_k\}_{k=1}^\infty$. Suppose that $\lim_{j \to \infty} x_{k_j} = x$. Since

$$\rho\big(f_{n_{k_j}}(x_{k_j}), f(x_{k_j})\big) \geqslant \varepsilon \qquad \forall\, j \in \mathbb{N}\,,$$

by the fact that $\lim\limits_{j\to\infty} f_{n_{k_j}}(x_{k_j}) = f(x)$ and that f is continuous at x, we obtain that

$$
\begin{aligned}
&\rho\big(f(x), f(x)\big) \\
&\qquad = \lim_{j\to\infty} \rho\big(f(x_{k_j}), f(x)\big) \geqslant \liminf_{j\to\infty} \Big[\rho\big(f_{n_{k_j}}(x_{k_j}), f(x_{k_j})\big) - \rho\big(f_{n_{k_j}}(x_{k_j}), f(x)\big)\Big] \\
&\qquad = \liminf_{j\to\infty} \rho\big(f_{n_{k_j}}(x_{k_j}), f(x_{k_j})\big) \geqslant \frac{\varepsilon}{2},
\end{aligned}
$$

a contradiction. □

Remark. Using the inequality

$$
\rho\big(f_k(x_k), f(x)\big) \leqslant \rho\big(f(x_k), f(x)\big) + \sup_{x\in A} \rho\big(f_k(x), f(x)\big),
$$

we find that if $\{f_k\}_{k=1}^\infty$ converges uniformly to a continuous function f, then $\lim\limits_{k\to\infty} f_k(x_k) = f(x)$ as long as $\lim\limits_{k\to\infty} x_k = x$. Together with the conclusion in 3, we conclude that

> Let (M,d), (N,ρ) be metric spaces, $K \subseteq M$ be a compact set, $f_k : K \to N$ be a function for each $k \in \mathbb{N}$, and $f : K \to N$ be continuous. The sequence $\{f_k\}_{k=1}$ converges uniformly to f if and only if $\lim\limits_{k\to\infty} f_k(x_k) = f(x)$ whenever sequence $\{x_k\}_{k=1}^\infty \subseteq K$ converges to x.

Problem 6. Let (M,d) be a metric space, $A \subseteq M$, (N,ρ) be a complete metric space, and $f_k : A \to N$ be a sequence of functions (not necessary continuous) which converges uniformly on A. Suppose that $a \in A'$ and

$$
\lim_{x\to a} f_k(x) = L_k
$$

exists for all $k \in \mathbb{N}$. Show that $\{L_k\}_{k=1}^\infty$ converges, and

$$
\lim_{x\to a}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\lim_{x\to a} f_k(x).
$$

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^\infty$ converges uniformly, there exists $N_1 > 0$ such that

$$
\rho\big(f_k(x), f_\ell(x)\big) < \frac{\varepsilon}{3} \quad \textit{whenever} \quad k,\ell \geqslant N_1 \text{ and } x \in A. \tag{$\star$}
$$

If $a \in \mathrm{cl}(A)$, then the inequality above implies that

$$
\rho(L_k, L_\ell) = \lim_{x\to a} \rho\big(f_k(x), f_\ell(x)\big) \leqslant \frac{\varepsilon}{3} < \varepsilon \quad \textit{whenever} \quad k,\ell \geqslant N_1;
$$

thus $\{L_k\}_{k=1}^\infty$ is a Cauchy sequence in (N,ρ). Therefore, $\{L_k\}_{k=1}^\infty$ converges. Suppose that $\lim\limits_{k\to\infty} L_k = L$ and $\{f_k\}_{k=1}^\infty$ converges uniformly to f. There exists $N_2 > 0$ such that $\rho(L_k, L) < \dfrac{\varepsilon}{3}$ whenever $k \geqslant N_2$. Moreover, passing to the limit as $\ell \to \infty$ in $(\star)$, we obtain that

$$
\rho\big(f_k(x), f(x)\big) \leqslant \frac{\varepsilon}{3} \quad \textit{whenever} \quad k \geqslant N_1 \text{ and } x \in A.
$$

Let $n = \max\{N_1, N_2\}$. Since $\lim\limits_{x\to a} f_n(x) = L_n$, there exists $\delta > 0$ such that

$$\rho\big(f_n(x), L_n\big) < \frac{\varepsilon}{3} \quad \textit{whenever} \quad x \in B(a,\delta) \cap A\backslash\{a\}\,.$$

Then if $x \in B(a,\delta) \cap A\backslash\{a\}$,

$$\rho\big(f(x), L\big) \leqslant \rho\big(f(x), f_n(x)\big) + \rho\big(f_n(x), L_n\big) + \rho(L_n, L) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.$$

Therefore, $\lim\limits_{x\to a} f(x) = L$ which shows that $\lim\limits_{x\to a}\lim\limits_{k\to\infty} f_k(x) = \lim\limits_{k\to\infty}\lim\limits_{x\to a} f_k(x)$. □

Problem 7. Prove the Dini theorem:

> Let K be a compact set, and $f_k : K \to \mathbb{R}$ be continuous for all $k \in \mathbb{N}$ such that $\{f_k\}_{k=1}$ converges pointwise to a continuous function $f : K \to \mathbb{R}$. Suppose that $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on K.

Hint: Mimic the proof of showing that $\{c_k\}_{k=1}^{\infty}$ converges to 0 in Lemma 6.64 in the lecture note.

Proof. Suppose the contrary that there exist $\varepsilon > 0$ such that

$$\limsup_{k\to\infty} \sup_{x\in K} \big|f_k(x) - f(x)\big| \geqslant 2\varepsilon\,.$$

Then there exists $1 \leqslant k_1 < k_2 < \cdots$ such that

$$\max_{x\in K} \big|f_{k_j}(x) - f(x)\big| = \sup_{x\in K} \big|f_{k_j}(x) - f(x)\big| > \varepsilon\,.$$

In other words, for some $\varepsilon > 0$ and strictly increasing sequence $\{k_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$,

$$F_j \equiv \big\{x \in K \,\big|\, f(x) - f_{k_j}(x) \geqslant \varepsilon\big\} \neq \varnothing \qquad \forall\, j \in \mathbb{N}\,.$$

Note that since $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$, $F_j \supseteq F_{j+1}$ for all $j \in \mathbb{N}$. Moreover, by the continuity of f_k and f, F_j is a closed subset of K; thus F_j is compact. Therefore, the nested set property for compact sets implies that $\bigcap_{j=1}^{\infty} F_j$ is non-empty. In other words, there exists $x \in K$ such that $f(x) - f_{k_j}(x) \geqslant \varepsilon$ for all $j \in \mathbb{N}$ which contradicts to the fact that $f_k \to f$ p.w. on K. □

Problem 8. Let (M,d) and (N,ρ) be metric spaces, $A \subseteq M$, and $f_k : A \to N$ be uniformly continuous functions, and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $f : A \to N$ on A. Show that f is uniformly continuous on A.

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f, there exists $N > 0$ such that

$$\rho\big(f_k(x), f(x)\big) < \frac{\varepsilon}{3} \quad \text{whenever} \quad k \geqslant N \text{ and } x \in A\,.$$

Since f_N is uniformly continuous, there exists $\delta > 0$ such that

$$\rho\big(f_N(x_1), f_N(x_2)\big) < \frac{\varepsilon}{3} \quad \text{whenever} \quad x_1, x_2 \in A \text{ and } d(x_1,x_2) < \delta\,.$$

Therefore, if $x_1, x_2 \in A$ satisfying $d(x_1,x_2) < \delta$, we have

$$\begin{aligned}\rho\big(f(x_1), f(x_2)\big) &\leqslant \rho\big(f(x_1), f_N(x_1)\big) + \rho\big(f_N(x_1), f_N(x_2)\big) + \rho\big(f_N(x_2), f(x_2)\big)\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,;\end{aligned}$$

thus f is uniformly continuous on A. □

Problem 9. Let (M,d) be a metric space, $(\mathcal{V},\|\cdot\|)$ be a norm space, $B\subseteq A\subseteq M$, $f_k:A\to\mathcal{V}$ be bounded for each $k\in\mathbb{N}$, and $\{g_n\}_{n=1}^\infty$ be the Cesàro mean of $\{f_k\}_{k=1}^\infty$; that is, $g_n=\frac{1}{n}\sum\limits_{k=1}^n f_k$. Show that if $\{f_k\}_{k=1}^\infty$ converges uniformly to f on B, then $\{g_n\}_{n=1}^\infty$ converges uniformly to f on B.

Proof. Let $\varepsilon>0$ be given. By the boundedness of f_k, for each $k\in\mathbb{N}$ there exists $M_k>0$ such that $\|f_k(x)\|\leqslant M_k$ for all $x\in B$ and $k\in\mathbb{N}$. Since $\{f_k\}_{k=1}^\infty$ converges uniformly to f on B, there exists $N_1>0$ such that

$$\|f_k(x)-f(x)\|<\frac{\varepsilon}{2}\qquad\forall\,k\geqslant N_1\text{ and }x\in B\,.$$

We note that the inequality above implies that $\|f(x)\|\leqslant M\equiv M_{N_1}+\varepsilon$ for all $x\in B$.

If $x\in B$, by the fact that

$$\sum_{k=1}^{N_1}\|f_k(x)-f(x)\|\leqslant\sum_{k=1}^{N_1}(M_k+M)<\infty\,,$$

we find that $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{N_1}\sup_{x\in B}\|f_k(x)-f(x)\|=0$; thus there exists $N_2>0$ such that

$$\frac{1}{n}\sum_{k=1}^{N_1}\|f_k(x)-f(x)\|<\frac{\varepsilon}{2}\qquad\text{whenever}\quad n\geqslant N_2\text{ and }x\in B\,.$$

Let $N=\max\{N_1,N_2\}$. Then if $n\geqslant N$ and $x\in B$,

$$\begin{aligned}\|g_n(x)-f(x)\|&=\Big|\frac{1}{n}\sum_{k=1}^n f_k(x)-f(x)\Big|\leqslant\frac{1}{n}\sum_{k=1}^{N_1}\|f_k(x)-f(x)\|+\frac{1}{n}\sum_{k=N_1}^{n}\|f_k(x)-f(x)\|\\&<\frac{\varepsilon}{2}+\frac{1}{n}\sum_{k=N_1}^{n}\frac{\varepsilon}{2}<\varepsilon\,;\end{aligned}$$

thus $\{g_n\}_{n=1}^\infty$ converges uniformly to f on B. $\square$