Exercise Problem Sets 3

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Problem 1. Let $A = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$, and $f : A \to \mathbb{R}$ be Riemann integrable. Show that the sets

$$\Big\{x \in [a, b] \,\Big|\, \underline{\int_c^d} f(x, y)dy \neq \overline{\int_c^d} f(x, y)dy\Big\} \quad \text{and} \quad \Big\{y \in [c, d] \,\Big|\, \underline{\int_a^b} f(x, y)dx \neq \overline{\int_a^b} f(x, y)dx\Big\}$$

have measure zero (in $\mathbb{R}^1$).

Proof. It suffices to show the former case. Define

$$g(x) = \overline{\int_c^d} f(x, y)\, dy - \underline{\int_c^d} f(x, y)\, dy\,.$$

Then g is non-negative on $[a, b]$. Moreover, the integrability of f implies that g is Rimenann integrable on $[a, b]$ and the Fubini theorem implies that $\displaystyle\int_a^b g(x)\, dx = 0$. Therefore, Part 2 of Theorem 6.45 shows that the set $\{x \in [a, b] \,|\, g(x) \neq 0\}$ has measure zero. □

Problem 2. Define a set $S \subseteq [0, 1] \times [0, 1]$ by

$$S = \Big\{\big(\frac{p}{m}, \frac{k}{m}\big) \in [0, 1] \times [0, 1] \,\Big|\, m, p, k \in \mathbb{N}\,, \gcd(m, p) = 1 \text{ and } 1 \leqslant k \leqslant m - 1\Big\}\,.$$

Show that

$$\int_0^1 \Big(\int_0^1 \mathbf{1}_S(x, y)\, dy\Big)dx = \int_0^1 \Big(\int_0^1 \mathbf{1}_S(x, y)\, dx\Big)dy = 0$$

but $\mathbf{1}_S$ is not Riemann integrable on $[0, 1] \times [0, 1]$.

Proof. Note that for each $x \in [0, 1]$, then $\mathbf{1}_S(x, y) \neq 0$ for only finitely many $y \in [0, 1]$. Therefore, for each $x \in [0, 1]$, $\mathbf{1}_S(x, \cdot)$ is Riemann integrable on $[0, 1]$ and

$$\int_0^1 \mathbf{1}_S(x, y)\, dy = 0\,.$$

Similarly, for each $y \in [0, 1]$, then $\mathbf{1}_S(x, y) \neq 0$ for only finitely many $x \in [0, 1]$; thus for each $y \in [0, 1]$, $\mathbf{1}_S(x, \cdot)$ is Riemann integrable on $[0, 1]$ and

$$\int_0^1 \mathbf{1}_S(x, y)\, dx = 0\,.$$

Therefore,

$$\int_0^1 \Big(\int_0^1 \mathbf{1}_S(x, y)\, dy\Big)dx = \int_0^1 \Big(\int_0^1 \mathbf{1}_S(x, y)\, dx\Big)dy = 0\,.$$

However, for each partition $\mathcal{P}$ of $[0, 1] \times [0, 1]$, we have $\Delta \cap S \neq \varnothing$ for all $\Delta \in \mathcal{P}$; thus $U(\mathbf{1}_S, \mathcal{P}) = 1$ for all partition $\mathcal{P}$ of $[0, 1] \times [0, 1]$. Therefore,

$$\overline{\int_{A\times B}} \mathbf{1}_S(x, y)\, d(x, y) = 1$$

which, by the Fubini Theorem, implies that $\mathbf{1}_S$ is not Riemann integrable on $[0, 1] \times [0, 1]$. □

Problem 3. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}\,, \\ -2^{2n+1} & \text{if } (x,y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n-1}, 2^{-n}),\ n \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

1. Show that $\displaystyle\int_0^1 f(x,y)\,dx = 0$ for all $y \in \big[0, \frac{1}{2}\big)$.
2. Show that $\displaystyle\int_0^1 f(x,y)\,dy = 0$ for all $x \in [0,1)$.
3. Justify if the iterated (improper) integrals $\displaystyle\int_0^1\int_0^1 f(x,y)dxdy$ and $\displaystyle\int_0^1\int_0^1 f(x,y)\,dydx$ are identical.

Proof. 1. Since $f(x,0) = 0$ for all $x \in [0,1]$, we have $\displaystyle\int_0^1 f(x,0)\,dx = 0$. Suppose that $y \in \big(0, \frac{1}{2}\big)$. Then $y \in [2^{-n}, 2^{-n+1})$ for a unique natural number $n \geqslant 2$. In this case,

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } x \in [2^{-n}, 2^{-n+1})\,, \\ -2^{2n-1} & \text{if } x \in [2^{-n+1}, 2^{-n+2})\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\begin{aligned} \int_0^1 f(x,y)\,dx &= \int_{[2^{-n},2^{-n+1})} 2^{2n}\,dx + \int_{[2^{-n+1},2^{-n+2})} -2^{2n-1}\,dx \\ &= 2^{2n}(2^{-n+1} - 2^{-n}) - 2^{2n-1}(2^{-n+2} - 2^{-n+1}) = 0\,. \end{aligned}$$

2. Since $f(0,y)$ for all $y \in [0,1]$, we have $\displaystyle\int_0^1 f(0,y)\,dy = 0$. Suppose tat $x \in (0,1)$. Then $x \in [2^{-n}, 2^{-n+1})$ for a unique $n \in \mathbb{N}$. In this case,

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } y \in [2^{-n}, 2^{-n+1}),\ n \in \mathbb{N}\,, \\ -2^{2n+1} & \text{if } y \in [2^{-n-1}, 2^{-n}),\ n \in \mathbb{N}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$\begin{aligned} \int_0^1 f(x,y)\,dy &= \int_{[2^{-n},2^{-n+1})} 2^{2n}\,dx + \int_{[2^{-n-1},2^{-n})} -2^{2n+1}\,dx \\ &= 2^{2n}(2^{-n+1} - 2^{-n}) - 2^{2n+1}(2^{-n} - 2^{-n-1}) = 0\,. \end{aligned}$$

3. By 2, we immediately conclude that

$$\int_0^1\int_0^1 f(x,y)\,dy\,dx = 0\,.$$

On the other hand, note that if $y \in \big[\frac{1}{2}, 1\big)$, then $f(x,y) = \begin{cases} 4 & \text{if } x \in \big[\frac{1}{2}, 1\big)\,, \\ 0 & \text{otherwise}\,, \end{cases}$ so that

$$\int_0^1 f(x,y)\,dx = \int_{\frac{1}{2}}^1 4\,dx = 2\,.$$

Therefore,

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^{\frac{1}{2}}\int_0^1 f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^1\int_0^1 f(x,y)\,dx\,dy = \int_{\frac{1}{2}}^1 2\,dy = 1$$

which shows that $\displaystyle\int_0^1\int_0^1 f(x,y)dxdy \neq \int_0^1\int_0^1 f(x,y)\,dydx$ for this particular f. □

Problem 4. Let $A = [a,b]$ be a closed interval in $\mathbb{R}$, and $f_k : A \to \mathbb{R}$ be a non-decreasing sequence (that is, $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$) of continuous functions such that

$$\lim_{k\to\infty} f_k(x) = f(x) \qquad \forall\, x \in A$$

for some continuous function $f : A \to \mathbb{R}$. Show that

$$\lim_{k\to\infty}\sup_{x\in A}\big|f_k(x) - f(x)\big| = 0\,.$$

If we do not assume that f is continuous or if A is replaced by other kind of intervals, is the conclusion still true?
Hint: Mimic the proof of Lemma 6.64 in the lecture note.

Proof. Suppose the contrary that there exist $\varepsilon > 0$ such that

$$\limsup_{k\to\infty}\sup_{x\in A}\big|f_k(x) - f(x)\big| \geqslant 2\varepsilon\,.$$

Then there exists $1 \leqslant k_1 < k_2 < \cdots$ such that

$$\max_{x\in A}\big|f_{k_j}(x) - f(x)\big| = \sup_{x\in A}\big|f_{k_j}(x) - f(x)\big| > \varepsilon\,.$$

In other words, for some $\varepsilon > 0$ and strictly increasing sequence $\{k_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$,

$$F_j \equiv \big\{x \in A \,\big|\, f(x) - f_{k_j}(x) \geqslant \varepsilon\big\} \neq \varnothing \qquad \forall\, j \in \mathbb{N}\,.$$

Note that since $f_k \leqslant f_{k+1}$ for all $k \in \mathbb{N}$, $F_j \supseteq F_{j+1}$ for all $j \in \mathbb{N}$. Moreover, by the continuity of f_k and f, F_j is a closed subset of A; thus F_j is compact. Therefore, the nested set property for compact sets (see Problem 4 of Exercise 9 from the previous semester) implies that $\bigcap_{j=1}^{\infty} F_j$ is non-empty. In other words, there exists $x \in A$ such that $f(x) - f_{k_j}(x) \geqslant \varepsilon$ for all $j \in \mathbb{N}$ which contradicts to the fact that $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$.

If f is not necessarily continuous, the conclusion is false. A counter-example is given as follows. Let $A = [0,1]$ and $f_k(x) = x^k$. Then we find that

$$\lim_{k\to\infty} f_k(x) = f(x) \qquad \forall\, x \in A\,,$$

where $f : [0,1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1)\,,\\ 1 & \text{if } x = 1\,. \end{cases}$$

Clearly f is not a continuous function on $[0,1]$. Moreover, we have

$$\sup_{x\in A}\big|f_k(x)-f(x)\big| = \sup_{x\in[0,1)}\big|f_k(x)\big| = \sup_{x\in[0,1)} x^k = 1$$

which shows that $\lim\limits_{k\to\infty}\sup\limits_{x\in A}\big|f_k(x)-f(x)\big| \neq 0$. □

Problem 5. Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous function, and $F(x)=\int_c^d f(x,y)\,dy$. Use the bounded convergence theorem to show that F is continuous on $[a,b]$.

Proof. Let $\{x_k\}_{k=1}^\infty \subseteq A$ be a sequence converging to $c\in[a,b]$, and define $g_n(y)=\sup\limits_{k\geqslant n} f(x_k,y)$. By the continuity of f, $\lim\limits_{n\to\infty} g_n(y)=\limsup\limits_{k\to\infty} f(x_k,y)=f(c,y)$ for all $y\in[c,d]$; thus $\{g_n\}_{n=1}^\infty$ converges pointwise to $f(a,y)$. Since $\{g_n\}_{n=1}^\infty$ is a decreasing sequence, Theorem **??** implies that

$$\lim_{n\to\infty}\underline{\int}_{[c,d]}\big(g_n(y)-f(a,y)\big)\,dy=0\,;$$

thus by the fact that $f(x_n,y)\leqslant g_n(y)$ for all $y\in[c,d]$;

$$\limsup_{n\to\infty}\int_c^d\big(f(x_n,y)-f(a,y)\big)\,dy \leqslant \lim_{n\to\infty}\underline{\int}_{[c,d]}\big(g_n(y)-f(a,y)\big)\,dy=0\,.$$

As a consequence, $\limsup\limits_{n\to\infty}F(x_n)\leqslant F(a)$. Since the sequence $\{x_n\}_{n=1}^\infty$ can be chosen arbitrarily, we conclude that $\limsup\limits_{x\to a}F(x)\leqslant F(a)$.

On the other hand, defining $h_n(y)=\inf\limits_{k\geqslant n} f(x_k,y)$, we have $h_n\leqslant h_{n+1}$ for all $n\in\mathbb{N}$ and $\{h_n\}_{n=1}^\infty$ converges pointwise to $f(a,y)$ on B; thus

$$\lim_{n\to\infty}\underline{\int}_{[c,d]}\big(f(a,y)-h_n(y)\big)\,dy=0\,.$$

By the fact that $h_n(y)\leqslant f(x_n,y)$ for all $y\in[c,d]$, we find that

$$\limsup_{n\to\infty}\int_c^d\big(f(a,y)-f(x_n,y)\big)\,dy \leqslant \lim_{n\to\infty}\underline{\int}_{[c,d]}\big(f(a,y)-h_n(y)\big)\,dy\leqslant 0\,;$$

thus $F(a)-\liminf\limits_{n\to\infty}F(x_n)\leqslant 0$. □

Problem 6 (The multiple integral version of Theorem 6.65 in the lecture note)**.** Let A be a closed rectangle in $\mathbb{R}^n$, and $f_k:A\to\mathbb{R}$ be a decreasing sequence of bounded functions. Show (without applying Theorem 6.69 and 6.70 in the lecture note) that if $\lim\limits_{k\to\infty}f_k(x)=0$ for all $x\in A$, then

$$\lim_{k\to\infty}\underline{\int}_A f_k(x)\,dx=0\,.$$

Conclude the Monotone Convergence Theorem (Theorem 6.69 in the lecture note) and the Bounded Convergence Theorem (Theorem 6.70 in the lecture note) using the this conclusion of convergence.

Proof. Let $\varepsilon > 0$ be given. Similar to the proof of Lemma 6.63 in the lecture, for each $k \in \mathbb{N}$ there exists a continuous function $g_k : A \to \mathbb{R}$ such that $0 \leqslant g_k \leqslant f_k$ and

$$\underline{\int_A} f_k(x)\,dx < \int_A g_k(x)\,dx + \frac{\varepsilon}{2^{k+1}}\,. \tag{0.1}$$

Define $h_k = \min\{g_1, \cdots, g_k\}$. Then h_k is continuous on A, $h_k \geqslant h_{k+1}$ (that is, $\{h_k\}_{k=1}^{\infty}$ is a decreasing sequence of funtions), $0 \leqslant h_k \leqslant g_k \leqslant f_k$ for all $k \in \mathbb{N}$, and $\lim\limits_{k\to\infty} h_k(x) = 0$ for all $x \in A$. Again, by Problem 5 (with A replaced by closed rectangle) we find that $\{h_k\}_{k=1}^{\infty}$ converges to the zero function in the following sense:

$$\lim_{k\to\infty} \sup_{x\in A} \big|h_k(x)\big| = 0\,;$$

thus there exists $N > 0$ such that

$$\int_A h_k(x)\,dx < \frac{\varepsilon}{4} \qquad \forall\, k \geqslant N\,. \tag{0.2}$$

On the other hand, for $1 \leqslant \ell \leqslant k$, $\max\{g_\ell, \cdots, g_k\} \leqslant \max\{f_\ell, \cdots, f_k\} = f_\ell$; thus

$$\int_A \big(\max\{g_\ell, \cdots, g_k\}(x) - g_\ell(x)\big)\,dx \leqslant \underline{\int_A} f_\ell(x)\,dx - \int_A g_\ell(x)\,dx < \frac{\varepsilon}{2^{\ell+1}}\,.$$

Moreover, for each $1 \leqslant j \leqslant k$ and $x \in A$,

$$\begin{aligned}
0 \leqslant g_k(x) &= g_j(x) + \big(g_k(x) - g_j(x)\big) \leqslant g_j(x) + \big(\max\big\{g_j(x), \cdots, g_k(x)\big\} - g_j\big)\\
&\leqslant g_j(x) + \sum_{\ell=1}^{k-1} \big(\max\big\{g_\ell, \cdots, g_k\big\}(x) - g_\ell(x)\big)\,,
\end{aligned}$$

so minimizing the right-hand side over all $1 \leqslant j \leqslant k$ implies that

$$0 \leqslant g_k(x) \leqslant h_k(x) + \sum_{\ell=1}^{k-1} \big(\max\{g_\ell, \cdots, g_k\}(x) - g_\ell(x)\big) \qquad \forall\, x \in A\,.$$

As a consequence,

$$0 \leqslant \int_A g_k(x)\,dx \leqslant \int_A h_k(x)\,dx + \sum_{\ell=1}^{k-1} \frac{\varepsilon}{2^{\ell+1}} \leqslant \int_A h_k(x)\,dx + \frac{\varepsilon}{2}\,;$$

thus (0.1) and (0.2) imply that

$$0 \leqslant \underline{\int_A} f_k(x)dx < \varepsilon \qquad \forall\, k \geqslant N\,.$$

Now suppose that $\{f_k\}_{k=1}^{\infty}$ is a monotone increasing sequence of Riemann integrable functions on A and for some Riemann integrable function f we have $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$. Define $g_k(x) = f(x) - f_k(x)$. Then $\{g_k\}_{k=1}^{\infty}$ is a decreasing sequence of bounded function and $\lim\limits_{k\to\infty} g_k(x) = 0$ for all $x \in A$; thus

$$\lim_{k\to\infty} \underline{\int_A} g_k(x)\,dx = 0\,.$$

Nevertheless, since $g_k = f - f_k$ and both f and f_k are Riemann integrable on A, we have

$$\underline{\int}_A g_k(x)\,dx = \int_A g_k(x)\,dx = \int_A f(x)\,dx - \int_A f_k(x)\,dx$$

so that we conclude that $\lim\limits_{k\to\infty} \int_A f_k(x)\,dx = \int_A f(x)\,dx$.

Now suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of Riemann integrable functions such that $\big|f_k(x)\big| \leqslant M$ for all $k \in \mathbb{N}$ and $x \in A$, and $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in A$. Define $g_k : A \to \mathbb{R}$ by $g_k(x) = \sup\limits_{\ell \geqslant k} \big|f_\ell(x) - f(x)\big|$. Then $\{g_k\}_{k=1}^{\infty}$ is a decreasing sequence of bounded functions and $\lim\limits_{k\to\infty} g_k(x) = 0$ for all $x \in A$. Therefore,

$$\lim_{k\to\infty} \underline{\int}_A g_k(x)\,dx = 0\,.$$

On the other hand, $\big|f_k(x) - f(x)\big| \leqslant g_k(x)$ for all $k \in \mathbb{N}$ and $x \in A$; thus the integrability of f_k and f implies that

$$\int_A \big|f_k(x) - f(x)\big|\,dx = \underline{\int}_A \big|f_k(x) - f(x)\big|\,dx \leqslant \underline{\int}_A g_k(x)\,dx$$

and the Sandwich lemma further shows hat

$$\lim_{k\to\infty} \int_A \big|f_k(x) - f(x)\big|\,dx = 0\,.$$

Therefore, $\lim\limits_{k\to\infty} \int_A f_k(x)\,dx = \int_A f(x)\,dx$. $\square$