Exercise Problem Sets 2

Mar. 02. 2023

Problem 1. Let $A \subseteq \mathbb{R}^n$ be an open bounded set with volume, and $f : A \to \mathbb{R}$ be continuous. Show that if $\int_B f(x)\,dx = 0$ for all subsets $B \subseteq A$ with volume, then $f = 0$.

Proof. Assume that for some $a \in A$, $f(a) \neq 0$. W.L.O.G. we can assume that $f(a) > 0$. By the continuity of f, there exists $\delta > 0$ such that

$$\big|f(x) - f(a)\big| < \frac{f(a)}{2} \quad \text{whenever} \quad x \in B(a,\delta) \cap A\,.$$

Since A is open, we can choose $0 < r < \delta$ such that $B(a,r) \subseteq B(a,\delta) \cap A$, and let B be a rectangle in $B(a,r)$ with sides parallel to the coordinate axes. Then B have volume and

$$\frac{f(a)}{2} < f(x) < \frac{3f(a)}{2} \quad \text{whenever} \quad x \in B\,.$$

This implies that $\int_B f(x)\,dx \geqslant \frac{f(a)}{2}\nu(B) > 0$, a contradiction. □

Problem 2. Prove the following statements.

1. The function $f(x) = \sin\frac{1}{x}$ is Riemann integrable on $(0,1)$.
2. Let $f : (0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \in \mathbb{Q},\ (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Then f is Riemann integrable on $(0,1]$. Find $\int_{(0,1]} f(x)dx$ as well.

3. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ is Riemann integrable. Then f^k（f 的 k 次方）is integrable for all $k \in \mathbb{N}$.

Proof. 1. Note that $(0,1)$ has volume, f is bounded on $(0,1)$ and f is continuous on $(0,1)$. Therefore, the Lebesgue Theorem (or its corollary) implies that f is Riemann integrable on $(0,1)$.

2. In Calculus we have shown that f is continuous on $\mathbb{Q}^\complement \cap (0,1]$ so that the collection of discontinuities of $\overline{f}^{(0,1]}$ is $\mathbb{Q} \cap (0,1]$. Since $\mathbb{Q} \cap (0,1]$ is countable, we find that the collection of discontinuities of $\overline{f}^{(0,1]}$ has measure zero. Therefore, f is Riemann integrable on $(0,1]$.

   Let $\mathcal{P}$ be a partition of $(0,1]$. Then $L(f,\mathcal{P}) = 0$ since

$$\inf_{x\in\Delta} \overline{f}^{(0,1]}(x) = 0 \qquad \forall\, \Delta \in \mathcal{P}\,.$$

   Therefore, $\underline{\int}_A f(x)\,dx = 0$; thus the fact that f is Riemann integrable on $(0,1]$ shows that $\int_{(0,1]} f(x)\,dx = 0$.

3. First we note that the fact that f is Riemann integrable on A implies that f is bounded on A. Therefore, f^k is bounded on A. Moreover, the Lebesgue Theorem implies that the collection D of discontinuities of $\overline{f}^A$ has measure zero. Since $\overline{f^k}^A = (\overline{f}^A)^k$, we find that the collection of discontinuities of $\overline{f^k}^A$ is exactly D; thus has measure zero. The Lebesgue Theorem then implies that f^k is Riemann integrable on A. □

Problem 3. Suppose that $f : [a,b] \to \mathbb{R}$ is Riemann integrable, and the set $\{x \in [a,b] \mid f(x) \neq 0\}$ has measure zero. Show that $\int_a^b f(x)\,dx = 0$.

Proof. First we note that for each $[c,d] \subseteq [a,b]$, then there exists $x \in [c,d]$ such that $f(x) = 0$ for otherwise $f(x) \neq 0$ for all $x \in [c,d]$ so that

$$[c,d] \subseteq \{x \in [a,b] \mid f(x) \neq 0\}$$

and this implies that $[c,d]$ is a set of measure zero, a contradiction to Corollary 6.25 in the lecture note. Therefore, $L(|f|,\mathcal{P}) = 0$ for all partitions $\mathcal{P}$ of $[a,b]$ which shows that $\underline{\int_a^b} f(x)\,dx = 0$. Since f is Riemann integrable on $[a,b]$, we conclude that $\int_a^b f(x)\,dx = 0$. □

Problem 4. Find an example that

$$\underline{\int_A} f(x)\,dx + \underline{\int_A} g(x)\,dx < \underline{\int_A} (f+g)(x)\,dx < \overline{\int_A} (f+g)(x)\,dx < \overline{\int_A} f(x)\,dx + \overline{\int_A} g(x)\,dx\,.$$

Solution. Let $f, g : [0,2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,2]\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}^\complement \cap [0,1]\,, \\ 0 & \text{otherwise}\,. \end{cases}\,.$$

Then for $A = [0,2]$,

$$\underline{\int_A} f(x)\,dx = \underline{\int_A} g(x)\,dx = 0\,, \quad \overline{\int_A} f(x)\,dx = 2 \quad \text{and} \quad \overline{\int_A} g(x)\,dx = 1\,.$$

Moreover,

$$(f+g)(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cup (\mathbb{Q} \cap [1,2])\,, \\ 0 & \text{otherwise.} \end{cases}$$

so that

$$\underline{\int_A} (f+g)(x)\,dx = 1 \qquad \text{and} \qquad \overline{\int_A} (f+g)(x)\,dx = 2\,.$$

Therefore, f and g satisfy the desired inequality.

Another example is given as follows: let $f, g : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 0 & \text{if } x \in \mathbb{Q}^\complement \cap [0,1]\,, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 2 & \text{if } x \in \mathbb{Q}^\complement \cap [0,1]\,, \end{cases}$$

Then

$$(f+g)(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]\,, \\ 2 & \text{if } x \in \mathbb{Q}^\complement \cap [0,1]\,, \end{cases}$$

so that we have $\underline{\int}_{[0,1]} f(x)\,dx = \underline{\int}_{[0,1]} g(x)\,dx = 0$, $\overline{\int}_{[0,1]} f(x)\,dx = \underline{\int}_{[0,1]} (f+g)(x)\,dx = 1$, and $\overline{\int}_{[0,1]} g(x)\,dx = \overline{\int}_{[0,1]} (f+g)(x)\,dx = 2$. □

Problem 5. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a bounded function. Show that if f is Riemann integrable on A, then $|f|$ is also Riemann integrable on A.

Proof. **Method 1**: Since f is Riemnn integrable on A, the Lebesgue Theorem implies that the collection of discontinuities of $\overline{f}^A$ has measure zero. Note that if $\overline{f}^A$ is continuous at $a \in A$, then $\overline{|f|}^A$ is also continuous at a since $\overline{|f|}^A = |\overline{f}^A|$. Therefore, the collection of discontinuities of $\overline{|f|}^A$ is a subset of a measure zero set, the collection of discontinuities of $\overline{f}^A$; thus the collection of discontinuities of $\overline{|f|}^A$ has measure zero. The Lebesgue Theorem then shows that $|f|$ is Riemann integrable on A.

Method 2: Let $\varepsilon > 0$ be given. Since f is Riemann integrable on A, by Riemann's condition there exists a partition $\mathcal{P}$ of A such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon\,.$$

Note that for each $\Delta \in P$,

$$\sup_{x\in\Delta} \big|\overline{f}^A(x)\big| - \inf_{x\in\Delta} \big|\overline{f}^A(x)\big| \leqslant \sup_{x\in\Delta} \overline{f}^A(x) - \inf_{x\in\Delta} \overline{f}^A(x)\,;$$

thus

$$\begin{aligned} U(|f|,\mathcal{P}) - L(|f|,\mathcal{P}) &= \sum_{\Delta\in\mathcal{P}} \Big(\sup_{x\in\Delta} \big|\overline{f}^A(x)\big| - \inf_{x\in\Delta} \big|\overline{f}^A(x)\big|\Big)\nu(\Delta) \\ &\leqslant \sum_{\Delta\in\mathcal{P}} \Big(\sup_{x\in\Delta} \overline{f}^A(x) - \inf_{x\in\Delta} \overline{f}^A(x)\Big)\nu(\Delta) = U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon\,. \end{aligned}$$

By Riemann's condition, we conclude that $|f|$ is Riemann integrable on A. □