Exercise Problem Sets 1

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Problem 1. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f : A \to \mathbb{R}$ be a function.

1. Show that if f is Riemann integrable on A, then f is bounded.
2. Show that if f is Darboux integrable on A, then f is bounded.

Note that we in some sense use these properties in the proof of the equivalence between the Riemann integrability and the Darboux integrability, so you'd better not use this equivalence in the proof.

Proof. 1. Since f is Riemann integrable on A, there exists $I \in \mathbb{R}$ and $\delta > 0$ such that if $\mathcal{P}$ is a partition of A satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ locates in $(\mathrm{I}-1, \mathrm{I}+1)$. Let $\mathcal{P} = \{\Delta_1, \Delta_2, \cdots, \Delta_N\}$ be a partition of A satisfying $\|\mathcal{P}\| < \delta$. For each $1 \leqslant k \leqslant N$, let c_k be the center of Δ_k. Then for each $1 \leqslant \ell \leqslant N$,

$$\mathrm{I} - 1 < \overline{f}^A(x)\nu(\Delta_\ell) + \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k) < \mathrm{I} + 1 \qquad \forall\, x \in \Delta_\ell$$

since $\overline{f}^A(x)\nu(\Delta_\ell) + \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)$ is a Riemann sum of f for $\mathcal{P}$. In particular,

$$\mathrm{I} - 1 < f(x)\nu(\Delta_\ell) + \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k) < \mathrm{I} + 1 \qquad \forall\, x \in \Delta_\ell \cap A\,.$$

which further implies that

$$\frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I} - 1 - \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] < f(x) < \frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I} + 1 - \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big]$$

Since f is real-valued, $\overline{f}^A(c_k)$ is a real number. The numbers M and m defined by

$$M \equiv \max\Big\{\frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I} + 1 - \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] \,\Big|\, 1 \leqslant \ell \leqslant N\Big\}\,,$$
$$m \equiv \min\Big\{\frac{1}{\nu(\Delta_\ell)}\Big[\mathrm{I} - 1 - \sum_{1\leqslant k\leqslant N, k\neq \ell} \overline{f}^A(c_k)\nu(\Delta_k)\Big] \,\Big|\, 1 \leqslant \ell \leqslant N\Big\}\,,$$

are both real numbers. Moreover, $m \leqslant f(x) \leqslant M$ for all $x \in A$; thus f is bounded.

2. Let $\mathcal{P}$ be a partition of A, and $\Delta \in \mathcal{P}$. Since f is real-valued, we must have

$$-\infty < \sup_{x\in\Delta} \overline{f}^A(x) \leqslant \infty \qquad \text{and} \qquad -\infty \leqslant \inf_{x\in\Delta} \overline{f}^A(x) < \infty\,.$$

The fact above implies that

(a) if f is not bounded from above, then $U(f, \mathcal{P}) = \infty$ for all partitions $\mathcal{P}$ of A;

(b) if f is not bounded from below, then $L(f, \mathcal{P}) = -\infty$ for all partitions $\mathcal{P}$ of A.

Therefore, if f is not bounded, either $\overline{\int_A} f(x)\,dx = \infty$ or $\underline{\int_A} f(x)\,dx = -\infty$; thus if f is Darboux integrable on A, then f must be bounded. $\square$

Problem 2. Let $A \subseteq \mathbb{R}^n$ be a bounded set, and $f, g : A \to \mathbb{R}$ be functions. Show that

$$\underline{\int_A} f(x)\,dx \leqslant \underline{\int_A} g(x)\,dx \qquad \text{and} \qquad \overline{\int_A} f(x)\,dx \leqslant \overline{\int_A} g(x)\,dx\,.$$

Proof. By the fact that $\overline{f}^A \leqslant \overline{g}^A$ on $\mathbb{R}^n$, we find that

$$U(f,\mathcal{P}) \leqslant U(g,\mathcal{P}) \quad \text{and} \quad L(f,\mathcal{P}) \leqslant L(g,\mathcal{P}) \qquad \forall \text{ partitions } \mathcal{P} \text{ of } A.$$

Since $\overline{\int_A} f(x)\,dx$ is a lower bound for $\big\{U(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$ and $\underline{\int_A} g(x)\,dx$ is an upper bound for $\big\{L(g,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$, we find that

$$\overline{\int_A} f(x)\,dx \leqslant U(f,\mathcal{P}) \leqslant U(g,\mathcal{P}) \;\text{ and }\; L(f,\mathcal{P}) \leqslant L(g,\mathcal{P}) \leqslant \underline{\int_A} g(x)\,dx \qquad \forall \text{ partitions } \mathcal{P} \text{ of } A.$$

The inequalities above shows that $\overline{\int_A} f(x)\,dx$ is a lower bound for $\big\{U(g,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$ and $\underline{\int_A} g(x)\,dx$ is an upper bound for $\big\{L(f,\mathcal{P}) \,\big|\, \mathcal{P} \text{ is a partition of } A\big\}$; thus we conclude that

$$\underline{\int_A} f(x)\,dx \leqslant \underline{\int_A} g(x)\,dx \qquad \text{and} \qquad \overline{\int_A} f(x)\,dx \leqslant \overline{\int_A} g(x)\,dx\,. \qquad \square$$

Problem 3. 1. Let $f : [0,1] \to \mathbb{R}$ be a bounded monotone function. Show that f is Riemann integrable on $[0,1]$.

2. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bounded function such that $f(x,y) \leqslant f(x,z)$ if $y < z$ and $f(x,y) \leqslant f(t,z)$ if $x < t$. In other words, $f(x,\cdot)$ and $f(\cdot,y)$ are both non-decreasing functions for fixed $x, y \in [0,1]$. Show that f is Riemann integrable on $[0,1] \times [0,1]$.

Proof. Let $\varepsilon > 0$ be given.

1. W.L.O.G., we can assume that f is increasing. Choose $n \in \mathbb{N}$ so that $\dfrac{f(1) - f(0)}{n} < \varepsilon$. Then if $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is a regular partition of $[0,1]$; that is, $x_k = \dfrac{(k-1)}{n}$, then the monotone

$$U(f,\mathcal{P}) = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

and

$$L(f,\mathcal{P}) = \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = \frac{1}{n}\sum_{k=1}^{n} f(x_{k-1})\,;$$

thus

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \frac{1}{n}\Big[\sum_{k=1}^{n} f(x_k) - \sum_{k=1}^{n} f(x_{k-1})\Big] = \frac{1}{n}\big[f(x_n) - f(x_0)\big] = \frac{f(1) - f(0)}{n} < \varepsilon\,.$$

Therefore, f is Riemann integrable on $[0,1]$ because of Riemann's condition.

2. Let $\mathcal{P}$ be a partition of $[0,1]\times[0,1]$. Then for $\Delta \in \mathcal{P}$,

$$\sup_{x\in\Delta} f(x) - \inf_{x\in\Delta} f(x) \leqslant f(\Delta_{ru}) - f(\Delta_{\ell b}),,$$

where Δ_{ur} and $\Delta_{b\ell}$ denote the up-right vertex and the bottom-left vertex of Δ. Therefore, with $\mathcal{P}_x = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ and $\mathcal{P}_y = \{0 = y_0 < y_1 < \cdots < y_n = 1\}$ denoting regular partitions of $[0,1]$ with $x_k = y_k = \dfrac{k-1}{n}$, we have

$$\begin{aligned}
U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \frac{1}{n^2}\sum_{k,\ell=1}^{n} f(x_k, y_\ell) - \frac{1}{n^2}\sum_{k,\ell=1}^{n} f(x_{k-1}, y_{\ell-1}) \\
&= \frac{1}{n^2}\Big[f(1,1) - f(0,0) + \sum_{k=1}^{n-1}\big(f(x_k,y_n) + f(x_n,y_k) - f(x_k,y_0) - f(x_0,y_k)\big)\Big]
\end{aligned}$$

Since $f(x,y) \leqslant f(x,z)$ if $y<z$ and $f(x,y) \leqslant f(t,z)$ if $x<t$, we have

$$f(x_k,y_n) - f(x_k,y_0) \leqslant f(1,1) - f(0,0) \quad \text{and} \quad f(x_n,y_k) - f(x_k,y_0) \leqslant f(1,1) - f(0,0)\,;$$

thus by choosing $n \gg 1$ so that $\dfrac{2}{n}\big[f(1,1) - f(0,0)\big] < \varepsilon$, we find that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) \leqslant \frac{1+2(n-1)}{n^2}\big[f(1,1) - f(0,0)\big] < \varepsilon\,.$$

Therefore, f is Riemann integrable on $[0,1]\times[0,1]$ because of Riemann's condition. □

Problem 4. Let $f, g : [a,b] \to \mathbb{R}$ be functions, where g is continuous, and f be non-negative, bounded, Riemann integrable on $[a,b]$. Show that fg is Riemann integrable.

Proof. Let $\varepsilon > 0$ be given, and $M > 0$ be an upper bounds of $f + |g|$; that is, $f(x) + |g(x)| \leqslant M$ for all $x \in [a,b]$. Since g is uniformly continuous on $[a,b]$, there exists $\delta > 0$ such that

$$\big|g(x) - g(y)\big| < \frac{\varepsilon}{8M(b-a)} \quad \text{whenever} \quad |x-y| < \delta$$

On the other hand, since f is Riemann integrable on $[a,b]$, by Riemann's condition there exists a partition $\mathcal{P}_1$ such that

$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) < \frac{\varepsilon}{2M}\,.$$

Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a refinement of $\mathcal{P}_1$ such that $\|\mathcal{P}\| < \delta$. For each $1 \leqslant k \leqslant n$, choose $\xi_k \in \Delta_k \equiv [x_{k-1}, x_k]$ such that

$$f(\xi_k)g(\xi_k) > \sup_{x\in\Delta_k}(fg)(x) - \frac{\varepsilon}{8(b-a)}$$

Then with $x_{k+\frac12}$ denoting the middle point of Δ_k, by the non-negativity of f we find that

$$\begin{aligned}
\sup_{x\in\Delta_k}(fg)(x) &< f(\xi_k)g(\xi_k) + \frac{\varepsilon}{8(b-a)} < f(\xi_k)\big[g(x_{k+\frac12}) + \frac{\varepsilon}{8M(b-a)}\big] + \frac{\varepsilon}{8(b-a)} \\
&\leqslant f(\xi_k)g(x_{k+\frac12}) + \frac{\varepsilon}{4(b-a)}\,.
\end{aligned}$$

Therefore,
$$U(fg,\mathcal{P}) \leqslant \sum_{k=1}^{n} f(\xi_k)g(x_{k+\frac{1}{2}})(x_k - x_{k-1}) + \frac{\varepsilon}{4}\,.$$
Similarly, if $\eta_k \in \Delta_k$ is chosen so that $f(\xi_k)g(\xi_k) < \inf\limits_{x\in\Delta_k}(fg)(x) + \dfrac{\varepsilon}{8(b-a)}$, then
$$L(fg,\mathcal{P}) \geqslant \sum_{k=1}^{n} f(\eta_k)g(x_{k+\frac{1}{2}})(x_k - x_{k-1}) - \frac{\varepsilon}{4}\,.$$
Therefore,
$$\begin{aligned} U(fg,\mathcal{P}) - L(fg,\mathcal{P}) &\leqslant \sum_{k=1}^{n} \big[f(\xi_k) - f(\eta_k)\big]g(x_{k+\frac{1}{2}})(x_k - x_{k-1}) + \frac{\varepsilon}{2} \\ &\leqslant \sum_{k=1}^{n} \big[\sup_{x\in\Delta_k} f(x) - \inf_{x\in\Delta_k} f(x)\big]M(x_k - x_{k-1}) + \frac{\varepsilon}{2} \\ &= M\big[U(f,\mathcal{P}) - L(f,\mathcal{P})\big] + \frac{\varepsilon}{2} < \varepsilon\,. \end{aligned}$$
Therefore, fg is Riemann integrable on $[a,b]$. □

Problem 5. Let $f : [a,b] \to \mathbb{R}$ be differentiable and assume that f' is Riemann integrable. Prove that $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$.

Hint: Use the Mean Value Theorem.

Proof. Let $\mathrm{I} = \displaystyle\int_a^b f'(x)\,dx$, and $\varepsilon > 0$ be given. Since f' is Riemann integrable on $[a,b]$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ locates in $(\mathrm{I}-\varepsilon, \mathrm{I}+\varepsilon)$. Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be such a partition. Then the Mean Value Theorem implies that for each $1 \leqslant k \leqslant n$ there exists $c_k \in (x_{k-1}, x_k)$ such that $f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1})$; thus
$$f(b) - f(a) = \sum_{k=1}^{n} \big[f(x_k) - f(x_{k-1})\big] = \sum_{k=1}^{n} f'(c_k)(x_k - x_{k-1})\,.$$
Note that the right-hand side is a Riemann sum of f for $\mathcal{P}$; thus $f(b) - f(a) \in (\mathrm{I}-\varepsilon, \mathrm{I}+\varepsilon)$ or
$$\mathrm{I} - \varepsilon < f(b) - f(a) < \mathrm{I} + \varepsilon\,.$$
Since $\varepsilon > 0$ is given arbitrarily, we conclude that $\mathrm{I} = f(b) - f(a)$. □

Problem 6. Suppose that $f : [a,b] \to \mathbb{R}$ is Riemann integrable, $m \leqslant f(x) \leqslant M$ for all $x \in [a,b]$, and $\varphi : [m,M] \to \mathbb{R}$ is continuous. Show that $\varphi \circ f$ is Riemann integrable on $[a,b]$.

Proof. Let $\varepsilon > 0$ be given. Since $\varphi : [m,M] \to \mathbb{R}$ is uniformly continuous, there exists $\delta > 0$ such that
$$\big|\varphi(y_1) - \varphi(y_2)\big| < \frac{\varepsilon}{2(b-a)} \quad \text{whenever } |y_1 - y_2| < \delta \text{ and } y_1, y_2 \in [m,M]\,.$$

Since f is Riemann integrable, there exists a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \frac{\varepsilon\delta}{4(\sup_{y\in[m,M]}|\varphi(y)|+1)}\,. \tag{0.1}$$

We claim that $U(\varphi\circ f,\mathcal{P}) - L(\varphi\circ f,\mathcal{P}) < \varepsilon$.

Let $m_i = \inf\limits_{x\in[x_{i-1},x_i]} f(x)$ and $M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$. Define

$$C_1 = \big\{1 \leqslant i \leqslant n \,\big|\, M_i - m_i < \delta\big\}\,, \qquad C_2 = \big\{1 \leqslant i \leqslant n \,\big|\, M_i - m_i \geqslant \delta\big\}\,.$$

Note that

$$\delta \sum_{i\in C_2} (x_i - x_{i-1}) \leqslant \sum_{i\in C_2} (M_i - m_i)(x_i - x_{i-1}) \leqslant U(f,\mathcal{P}) - L(f,\mathcal{P})\,;$$

thus (0.1) implies that

$$\sum_{i\in C_2} (x_i - x_{i-1}) < \frac{\varepsilon}{4(\sup_{y\in[m,M]}|\varphi(y)|+1)}\,.$$

Therefore,

$$\begin{aligned}
U(\varphi\circ f,\mathcal{P}) - L(\varphi\circ f,\mathcal{P}) &= \sum_{i=1}^n \Big[\sup_{x\in[x_{i-1},x_i]}(\varphi\circ f)(x) - \inf_{x\in[x_{i-1},x_i]}(\varphi\circ f)(x)\Big](x_i - x_{i-1}) \\
&= \sum_{i=1}^n \Big[\sup_{y\in[m_i,M_i]}\varphi(y) - \inf_{y\in[m_i,M_i]}\varphi(y)\Big](x_i - x_{i-1}) \\
&= \Big(\sum_{i\in C_1} + \sum_{i\in C_2}\Big)\Big[\sup_{y\in[m_i,M_i]}\varphi(y) - \inf_{y\in[m_i,M_i]}\varphi(y)\Big](x_i - x_{i-1}) \\
&\leqslant \sum_{i\in C_1} \frac{\varepsilon}{2(b-a)}(x_i - x_{i-1}) + 2\sup_{y\in[m,M]}|\varphi(y)| \sum_{i\in C_2}(x_i - x_{i-1}) \\
&\leqslant \frac{\varepsilon}{2} + \frac{2\sup_{y\in[m,M]}|\varphi(y)|\varepsilon}{4(\sup_{y\in[m,M]}|\varphi(y)|+1)} < \varepsilon\,.
\end{aligned}$$

Therefore, $\varphi\circ f$ is Riemann integrable on $[a,b]$. □

Problem 7. For a function $f:[a,b]\to\mathbb{R}$, define the ***total variation*** of f on $[a,b]$ by

$$V_a^b(f) = \sup\Big\{\sum_{k=1}^n \big|f(x_k) - f(x_{k-1})\big| \,\Big|\, \{a = x_0 < \cdots < x_n = b\} \text{ is a partition of } [a,b]\Big\}\,.$$

Sometimes $V_a^b(f)$ is written as $\|f\|_{\mathrm{TV}([a,b])}$.

A function $f:[a,b]$ is said to have ***bounded variation*** on $[c,d]$ or be ***of bounded variation*** on $[c,d]$, where $[c,d]\subseteq[a,b]$, if $V_c^d(f) < \infty$. Complete the following.

1. Let $\mathrm{BV}([a,b]) = \big\{f:[a,b]\to\mathbb{R} \,\big|\, V_a^b(f) < \infty\big\}$, called the space of functions of bounded variation (on $[a,b]$). Show that $\mathrm{BV}([a,b])$ is a vector space.

2. Is V_a^b a norm on $\mathrm{BV}([a,b])$; that is, does $\|\cdot\|_{\mathrm{TV}([a,b])} : \mathrm{BV}([a,b])\to\mathbb{R}$ defined by $\|f\| \equiv V_a^b(f)$ satisfy Definition **??**?

3. Recall that $\mathscr{C}^1([a,b];\mathbb{R}) \equiv \{f : [a,b] \to \mathbb{R} \,|\, f' \text{ is continuous on } [a,b]\}$. Show that if $f \in \mathscr{C}^1([a,b];\mathbb{R})$, then f is of bounded variation.

4. Show that if $f \in \mathscr{C}^1([a,b];\mathbb{R})$, then $V_a^b(f) = \displaystyle\int_a^b |f'(x)|\,dx$.

5. Show that if $V_a^b(f) < \infty$ (f is not necessarily differentiable everywhere), then

$$V_a^b(f) = \sup\Big\{ \int_a^b f(x)\phi'(x)\,dx \,\Big|\, \phi \in \mathscr{C}^1([a,b];\mathbb{R}), |\phi(x)| \leqslant 1 \text{ for all } x \in [a,b], \phi(a) = \phi(b) = 0 \Big\}.$$

Proof. For a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$, define

$$V(f,\mathcal{P}) = \sum_{k=1}^n |f(x_k) - f(x_{k-1})|.$$

We note that the triangle inequality implies that

$$V(f,\mathcal{P}) \leqslant V(f,\mathcal{P}') \quad \text{whenever} \quad \mathcal{P}' \text{ is a refinement of } \mathcal{P}. \tag{0.2}$$

1. Let $f, g \in \mathrm{BV}([a,b])$, $c \in \mathbb{R}$, and $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Then

$$\begin{aligned} V(cf+g,\mathcal{P}) &= \sum_{k=1}^n \big|(cf+g)(x_k) - (cf+g)(x_{k-1})\big| \\ &\leqslant \sum_{k=1}^n \Big[|c|\big|f(x_k) - f(x_{k-1})\big| + \big|g(x_k) - g(x_{k-1})\big|\Big] \\ &= |c|V(f,\mathcal{P}) + V(g,\mathcal{P}) \leqslant |c|V_a^b(f) + V_a^b(g). \end{aligned}$$

Therefore, $V_a^b(cf+g) \leqslant |c|V_a^b(f) + V_a^b(g) < \infty$ which shows that $cf+g \in \mathrm{BV}([a,b])$. Therefore, $\mathrm{BV}([a,b])$ is a vector space.

2. V_a^b is not a norm since any constant function has zero variation. This violates property (b) in the definition of norms.

3. Suppose that f is continuously differentiable on $[a,b]$. By the Extreme Value Theorem, $\sup\limits_{x\in[a,b]} |f'(x)| < \infty$. Therefore, for each partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, the Mean Value Theorem implies that

$$\begin{aligned} V(f,\mathcal{P}) &= \sum_{k=1}^n |f(x_k) - f(x_{k-1})| \leqslant \sum_{k=1}^n \sup_{x\in[x_{k-1},x_k]} |f'(x)|(x_k - x_{k-1}) \\ &\leqslant \sup_{x\in[a,b]} |f'(x)| \sum_{k=1}^n (x_k - x_{k-1}) = (b-a) \sup_{x\in[a,b]} |f'(x)| < \infty; \end{aligned}$$

thus $f \in \mathrm{BV}([a,b])$.

4. Suppose that f is continuously differentiable on $[a,b]$. Then f' is continuous on $[a,b]$; thus $|f'|$ is also continuous on $[a,b]$. Therefore, $I = \displaystyle\int_a^b \big|f'(x)\big|\,dx$ exists. Next we show that $V_a^b(f) = I$. Let $\varepsilon > 0$. By the definition of total variation, there exists a partition $\mathcal{P}_1$ of $[a,b]$ such that

$$V_a^b(f) - \frac{\varepsilon}{2} < V(f,\mathcal{P}_1)\,.$$

By the definition of integrals, there exists a partition $\mathcal{P}_2$ of $[a,b]$ such that

$$U\big(|f'|,\mathcal{P}_2\big) < I + \frac{\varepsilon}{2}\,.$$

Let $\mathcal{P}_3 = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. By the Mean Value Theorem, for each $1 \leqslant k \leqslant n$ there exists $\xi_k \in (x_{k-1}, x_k)$ such that

$$f(x_k) - f(x_{k-1}) = f'(\xi_k)(x_k - x_{k-1})\,;$$

thus (0.2) implies that

$$\begin{aligned} V_a^b(f) - \frac{\varepsilon}{2} &< V(f,\mathcal{P}_3) = \sum_{k=1}^n \big|f(x_k) - f(x_{k-1})\big| = \sum_{k=1}^n \big|f'(\xi_k)\big|(x_k - x_{k-1}) \\ &\leqslant \sum_{k=1}^n \sup_{x\in[x_{k-1},x_k]} \big|f'(x)\big|(x_k - x_{k-1}) = U\big(|f'|,\mathcal{P}_3\big) \leqslant U\big(|f'|,\mathcal{P}_2\big) < I + \frac{\varepsilon}{2}\,. \end{aligned}$$

Therefore,

$$V_a^b(f) < I + \varepsilon\,. \tag{0.3}$$

On the other hand, by the uniform continuity, there exists $\delta > 0$ such that

$$\Big|\big|f'(x)\big| - \big|f'(y)\big|\Big| < \frac{\varepsilon}{2(b-a)} \qquad \text{whenever} \quad |x-y| < \delta \text{ and } x,y \in [a,b]\,.$$

Let $\mathcal{P}_4 = \{a = y_0 < y_1 < \cdots < y_m = b\}$ be a refinement of $\mathcal{P}_2$ such that $\|\mathcal{P}_2\| < \delta$. The Mean Value Theorem implies that for each $1 \leqslant k \leqslant m$, there exists $\eta_k \in (y_{k-1}, y_k)$ such that

$$f(y_k) - f(y_{k-1}) = f'(\eta_k)(y_k - y_{k-1})\,.$$

Then for each $1 \leqslant k \leqslant m$,

$$\sup_{y\in[y_{k-1},y_k]} \big|f'(y)\big| \leqslant \big|f'(\eta_k)\big| + \frac{\varepsilon}{2(b-a)}\,.$$

The inequality above further implies that

$$\begin{aligned} I \leqslant U\big(|f'|,\mathcal{P}_4\big) &= \sum_{k=1}^m \sup_{y\in[y_{k-1},y_k]} \big|f'(y)\big|(y_k - y_{k-1}) \\ &\leqslant \sum_{k=1}^m \Big(\big|f'(\eta_k)\big| + \frac{\varepsilon}{2(b-a)}\Big)(y_k - y_{k-1}) \leqslant \sum_{k=1}^n \big|f(y_k) - f(y_{k-1})\big| + \frac{\varepsilon}{2} \\ &< V_a^b(f) + \varepsilon\,. \end{aligned}$$

Therefore, together with (0.3), we conclude that

$$\big|V_a^b(f) - I\big| < \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrary, we find that $V_a^b(f) = I$. $\square$