Exercise Problem Sets 12

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Problem 1. Check if the following functions on uniformly continuous.

1. $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = \sin\log x$.
2. $f : (0,1) \to \mathbb{R}$ defined by $f(x) = x\sin\dfrac{1}{x}$.
3. $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$.
4. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x^2)$.
5. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos^3 x$.
6. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\sin x$.

Problem 2. 1. Find all positive numbers a and b such that the function $f(x) = \dfrac{\sin(x^a)}{1+x^b}$ is uniformly continuous on $[0,\infty)$.

2. Find all positive numbers a and b such that the function $f(x,y) = |x|^a|y|^b$ is uniformly continuous on $\mathbb{R}^2$.

Problem 3. Show that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ defined by $f(x,y) = \dfrac{\sqrt{x}}{1+x^2y^2}$ is uniformly continuous on its domain.

Proof. Let $\varepsilon > 0$ be given. Choose $N > 0$ so that $\dfrac{8}{4+\varepsilon^2N^2} < \varepsilon$. Then

1. if $(x,y) \in \big[0,\dfrac{\varepsilon^2}{4}\big]\times[-N,N]^{\complement}$, we have $\big|f(x,y)\big| \leqslant \sqrt{x} < \dfrac{\varepsilon}{2}$.
2. if $(x,y) \in \big[\dfrac{\varepsilon^2}{4},1\big]\times[-N,N]^{\complement}$, we have $\big|f(x,y)\big| \leqslant \dfrac{1}{1+\frac{\varepsilon^2}{4}N^2} = \dfrac{4}{4+\varepsilon^2N^2} < \dfrac{\varepsilon}{2}$.

Therefore,

$$\big|f(x,y)\big| \leqslant \frac{\varepsilon}{2} \qquad \forall\,(x,y)\times[0,1]\times[-N,N]^{\complement}\,.$$

Since $[0,1]\times[-2N,2N]$ is compact, the continuity of f implies that f is uniformly continuous on $[0,1]\times[0,2N]$; thus there exists $\delta_1 > 0$ such that

$$\big|f(x_1,y_1)-f(x_2,y_2)\big| < \varepsilon \quad \forall\,\big|(x_1,y_1)-(x_2,y_2)\big| < \delta_1 \text{ and } x_1,x_2\in[0,1], y_1,y_2\in[-2N,2N]\,.$$

Define $\delta = \min\{\delta_1, N\}$. If $(x_1,y_1),(x_2,y_2)\in[0,1]\times\mathbb{R}$ and $\big|(x_1,y_1)-(x_2,y_2)\big| < \delta$, then either (x_1,y_1), (x_2,y_2) belongs to $[0,1]\times[-2N,2N]$ or (x_1,y_1), (x_2,y_2) belongs to $[0,1]\times[-N,N]^{\complement}$. Therefore,

$$\big|f(x_1,y_1)-f(x_2,y_2)\big| < \varepsilon \quad \forall\,(x_1,y_1),(x_2,y_2)\in[0,1]\times\mathbb{R} \text{ and } \big|(x_1,y_1)-(x_2,y_2)\big| < \delta\,. \qquad \square$$

Problem 4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and $\lim\limits_{|x|\to\infty} f(x) = b$ exists for some $b\in\mathbb{R}^m$. Show that f is uniformly continuous on $\mathbb{R}^n$.

Proof. Let $\varepsilon > 0$ be given. By the fact that $\lim\limits_{|x|\to\infty} f(x) = b$, there exists $M > 0$ such that

$$\|f(x) - b\|_{\mathbb{R}^m} < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x\|_{\mathbb{R}^n} \geqslant M\,.$$

By the Heine-Borel Theorem, $B[0, M+1]$ is compact; thus f is uniformly continuous on $B[0, M+1]$ and there exists $\delta \in \big(0, \frac{1}{2}\big)$ such that

$$\|f(x) - f(y)\| < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x - y\|_{\mathbb{R}^n} < \delta \text{ and } x, y \in B[0, M+1]\,. \tag{$\star$}$$

Therefore, for $x, y \in \mathbb{R}^n$ satisfying $\|x - y\| < \delta$,

1. if $x, y \in B[0, M+1]$, then ($\star$) implies that
$$\|f(x) - f(y)\|_{\mathbb{R}^m} < \varepsilon\,.$$
2. if $x \notin B[0, M+1]$ or $y \notin B[0, M+1]$, then $x, y \in B[0, M]^\complement$ which implies that
$$\|f(x) - f(y)\|_{\mathbb{R}^m} \leqslant \|f(x)\|_{\mathbb{R}^m} + \|f(y)\|_{\mathbb{R}^m} < \varepsilon\,.$$

□

Problem 5. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous. Show that there exists $a > 0$ and $b > 0$ such that $\|f(x)\|_{\mathbb{R}^m} \leqslant a\|x\|_{\mathbb{R}^n} + b$.

Proof. Since f is uniformly continuous on $\mathbb{R}^n$, there exists $\delta > 0$ such that

$$\big\|f(x) - f(y)\big\|_{\mathbb{R}^n} < 1 \quad \text{whenever} \quad \|x - y\|_{\mathbb{R}^n} < \delta\,.$$

For a given $x \in \mathbb{R}^n$, let $N \in \mathbb{N}$ such that $\dfrac{\|x\|_{\mathbb{R}^n}}{\delta} < N \leqslant \dfrac{\|x\|_{\mathbb{R}^n}}{\delta} + 1$. For each $k \in \mathbb{N}$, define points x_k by $x_k \equiv \dfrac{kx}{N}$. Then $\{x_k\}_{k=0}^{\infty}$ satisfies that

$$\|x_k - x_{k-1}\|_{\mathbb{R}^m} = \frac{\|x\|_{\mathbb{R}^n}}{N} < \delta \qquad \forall\, k \in \mathbb{N}$$

which further implies that

$$\|f(x_k) - f(x_{k-1})\|_{\mathbb{R}^m} < 1 \qquad \forall\, k \in \mathbb{N}\,.$$

Therefore,

$$\begin{aligned}
\|f(x)\|_{\mathbb{R}^m} &\leqslant \|f(x) - f(0)\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} \leqslant \sum_{k=1}^{N} \|f(x_k) - f(x_{k-1})\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} \\
&\leqslant N + \|f(0)\|_{\mathbb{R}^m} \leqslant \frac{1}{\delta}\|x\|_{\mathbb{R}^m} + \|f(0)\|_{\mathbb{R}^m} + 1\,;
\end{aligned}$$

thus $a = \dfrac{1}{\delta}$ and $b = \|f(0)\|_{\mathbb{R}^m} + 1$ verify the inequality $\|f(x)\|_{\mathbb{R}^m} \leqslant a\|x\|_{\mathbb{R}^n} + b$. □

Problem 6. Let $f(x) = \dfrac{q(x)}{p(x)}$ be a rational function define on $\mathbb{R}$, where p and q are two polynomials. Show that f is uniformly continuous on $\mathbb{R}$ if and only if the degree of q is not more than the degree of p plus 1.

Proof. Note that if f is defined on $\mathbb{R}$, then $p(x) \neq 0$ for all $x \in \mathbb{R}$. By Problem 5, there exist $a, b > 0$ such that

$$\left|\frac{q(x)}{p(x)}\right| \leqslant a|x| + b \qquad \forall\, x \in \mathbb{R}\,.$$

Therefore, $|q(x)| \leqslant |p(x)|(a|x| + b)$ for all $x \in \mathbb{R}$, and this inequality above can be true if and only if the degree of q is not more than the degree of p plus 1. □

Problem 7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function; that is, there exists $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$ (and f is continuous). Show that f is uniformly continuous on $\mathbb{R}$.

Proof. Let $p > 0$ be such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$, and $\varepsilon > 0$ be given. Since f is uniformly continuous on $[-p, p]$, there exists $\delta \in (0, p)$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2} \quad \text{whenever} \quad |x - y| < \delta \text{ and } x, y \in [-p, p]\,.$$

Therefore, if $|x-y| < \delta$, we must have $x, y \in [kp-p, kp+p]$ for some $k \in \mathbb{Z}$ so that $x-kp, y-kp \in [-p, p]$ which, together with the fact that $|(x - kp) - (y - kp)| = |x - y| < \delta$, implies that

$$\big|f(x) - f(y)\big| = \big|f(x - kp) - f(y - kp)\big| < \varepsilon\,.$$ □

Problem 8. Let $(a, b) \subseteq \mathbb{R}$ be an open interval, and $f : (a, b) \to \mathbb{R}^m$ be a function. Show that the following three statements are equivalent.

1. f is uniformly continuous on (a, b).
2. f is continuous on (a, b), and both limits $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ exist.
3. For all $\varepsilon > 0$, there exists $N > 0$ such that $\big|f(x) - f(y)\big| < \varepsilon$ whenever $\left|\dfrac{f(x) - f(y)}{x - y}\right| > N$ and $x, y \in (a, b)$, $x \neq y$.

Proof. First we note that 1 and 2 are equivalent since

1. if f is uniformly continuous on (a, b), then there is a unique continuous extension g of f on $[a, b]$; thus $\lim\limits_{x\to a^+} g(x) = g(a)$ and $\lim\limits_{x\to b^-} g(x) = g(b)$ exists, and 2 holds since $\lim\limits_{x\to a^+} g(x) = \lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} g(x) = \lim\limits_{x\to b^-} f(x)$.
2. if $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ exists, we define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x)$ for $x \in (a, b)$ and $g(a)$, $g(b)$ are respectively the limit of f at a, b. Then g is continuous on $[a, b]$; thus the compactness of $[a, b]$ shows that g is uniformly continuous on $[a, b]$. In particular, g is uniformly continuous on (a, b) which is the same as saying that f is uniformly continuous on (a, b).

Next we prove that 1 and 3 are equivalent.

"1 ⇒ 3" Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in (a,b)$ such that

$$x_n \neq y_n\,,\quad \big|f(x_n) - f(y_n)\big| \geqslant \varepsilon \quad \text{but} \quad \Big|\frac{f(x_n) - f(y_n)}{x_n - y_n}\Big| > n \qquad \forall\, n \in \mathbb{N}\,.$$

By the Bolzano-Weierstrass Theorem/Property, there exist convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ and $\{y_{n_j}\}_{j=1}^{\infty}$ with limit x and y. Since $x_n, y_n \in (a,b)$ for all $n \in \mathbb{N}$, we must have $x, y \in [a,b]$. If $x = y$, then $|x_n - y_n| \to 0$ as $n \to \infty$; thus the uniform continuity of f on (a,b) implies that $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$ which contradicts to the fact that $|f(x_n) - f(y_n)| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. Therefore, $x \neq y$ which further shows that the limit

$$\lim_{n\to\infty} \Big|\frac{f(x_n) - f(y_n)}{x_n - y_n}\Big|$$

exists since the limit $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ both exist and $\lim\limits_{n\to\infty}(x_n - y_n) = x - y \neq 0$. This is a contradiction to that $\Big|\dfrac{f(x_n) - f(y_n)}{x_n - y_n}\Big| > n$ for all $n \in \mathbb{N}$.

"3 ⇒ 1" Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n, y_n \in (a,b)$ satisfying $|x_n - y_n| < \dfrac{1}{n}$ but $|f(x_n) - f(y_n)| \geqslant \varepsilon$. For this $\varepsilon > 0$, by assumption there exists $N > 0$ such that

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever} \qquad \Big|\frac{f(x) - f(y)}{x - y}\Big| > N \text{ and } x, y \in (a,b), x \neq y\,.$$

Since $|f(x_n) - f(y_n)| \geqslant \varepsilon$, we must have $x_n \neq y_n$; thus the fact that $x_n, y_n \in (a,b)$ implies that

$$\Big|\frac{f(x_n) - f(y_n)}{x_n - y_n}\Big| \leqslant N \qquad \forall\, n \in \mathbb{N}\,.$$

This contradicts to the fact that $|x_n - y_n| < \dfrac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon$. □

Problem 9. Suppose that $f : [a,b] \to \mathbb{R}$ is ***Hölder continuous with exponent*** α; that is, there exist $M > 0$ and $\alpha \in (0,1]$ such that

$$|f(x) - f(y)| \leqslant M|x - y|^{\alpha} \qquad \forall\, x, y \in [a,b]\,.$$

Show that f is uniformly continuous on $[a,b]$. Show that $f : [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$.

Proof. Let $\varepsilon > 0$ be given. Define $\delta = M^{-\frac{1}{\alpha}}\varepsilon^{\frac{1}{\alpha}}$. Then $\delta > 0$. Moreover, if $|x - y| < \delta$ and $x, y \in [a,b]$,

$$|f(x) - f(y)| \leqslant M|x - y|^{\alpha} < M\delta^{\alpha} = \varepsilon\,.$$

Therefore, f is uniformly continuous on $[a,b]$.

Next we show that $f(x) = \sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$. Note that if $x, y \geqslant 0$ and $x \neq y$,

$$\frac{|\sqrt{x} - \sqrt{y}|}{|x - y|^{\frac{1}{2}}} = \frac{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|}{|x - y|^{\frac{1}{2}}|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|^{\frac{1}{2}}}{|\sqrt{x} + \sqrt{y}|} \leqslant \frac{\sqrt{x} + \sqrt{y}}{|\sqrt{x} + \sqrt{y}|} \leqslant 1\,;$$

thus

$$|\sqrt{x}-\sqrt{y}| \leqslant |x-y|^{\frac{1}{2}} \qquad \forall\, x,y \geqslant 0 \text{ and } x \neq y\,.$$

which implies that $f(x)=\sqrt{x}$ is Hölder continuous with exponent $\dfrac{1}{2}$ on $[0,\infty)$. □

Problem 10. A function $f : A \times B \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^p$, is said to be separately continuous if for each $x_0 \in A$, the map $g(y) = f(x_0, y)$ is continuous and for $y_0 \in B$, $h(x) = f(x, y_0)$ is continuous. f is said to be continuous on A uniformly with respect to B if

$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big\|f(x,y) - f(x_0,y)\big\|_2 < \varepsilon \quad \text{whenever} \quad \|x - x_0\|_2 < \delta \text{ and } y \in B\,.$$

Show that if f is separately continuous and is continuous on A uniformly with respect to B, then f is continuous on $A \times B$.

Proof. Let $\varepsilon > 0$, and $(a,b) \in A \times B$ be given. By assumption there exists $\delta_1 > 0$ such that

$$\big\|f(x,y) - f(a,y)\big\|_2 < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x-a\|_2 < \delta_1 \text{ and } x \in A\,, y \in B\,.$$

Since f is separately continuous, there exists $\delta_2 > 0$ such that

$$\big\|f(a,y) - f(a,b)\big\|_2 < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|y-b\|_2 < \delta_2 \text{ and } y \in B\,.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then if $\|(x,y)-(a,b)\|_2 < \delta$, we must have $\|x-a\|_2 < \delta_1$ and $\|y-b\|_2 < \delta_2$ so that

$$\begin{aligned} \|f(x,y) - f(a,b)\|_2 &= \|f(x,y) - f(a,y) + f(a,y) - f(a,b)\|_2 \\ &\leqslant \|f(x,y) - f(a,y)\|_2 + \|f(a,y) - f(a,b)\|_2 < \varepsilon \end{aligned}$$

which shows that f is continuous at (a,b). □

Problem 11. Let (M,d) be a metric space, $A \subseteq M$, and $f,g : A \to \mathbb{R}$ be uniformly continuous on A. Show that if f and g are bounded, then fg is uniformly continuous on A. Does the conclusion still hold if f or g is not bounded?

Proof. Let $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ be sequences in A satisfying that $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$. Suppose that $|f(x)| \leqslant M$ and $|g(x)| \leqslant M$ for all $x \in A$. Then

$$\begin{aligned} \big|f(x_n)g(x_n) - f(y_n)g(y_n)\big| &= \big|f(x_n)g(x_n) - f(x_n)g(y_n) + f(x_n)g(y_n) - f(y_n)g(y_n)\big| \\ &\leqslant |f(x_n)|\big|g(x_n) - g(y_n)\big| + |g(y_n)|\big|f(x_n) - f(y_n)\big| \\ &\leqslant M\big(\big|f(x_n) - f(y_n)\big| + \big|g(x_n) - g(y_n)\big|\big)\,; \end{aligned}$$

thus the uniform continuity of f and g, together with the Sandich Lemma, implies that

$$\lim_{n\to\infty} \big|f(x_n)g(x_n) - f(y_n)g(y_n)\big| = 0\,.$$

Therefore, fg is uniformly continuous on A.

When the boundedness is removed from the condition, the product of f and g might not be uniformly continuous. For example, $f(x) = g(x) = x$ are continuous on $\mathbb{R}$, but $(fg)(x) = x^2$ is no uniformly continuous on $\mathbb{R}$ (from an example in class). □