Exercise Problem Sets 11

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Problem 1. Complete the following.

1. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and $B \subseteq \mathbb{R}^n$ is bounded, then $f(B)$ is bounded.

2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subseteq \mathbb{R}$ is compact, is $f^{-1}(K)$ necessarily compact?

3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $C \subseteq \mathbb{R}$ is connected, is $f^{-1}(C)$ necessarily connected?

Solution. 1. Since B is bounded, $\bar{B}$ is closed and bounded; thus the Heine-Borel Theorem implies that $\bar{B}$ is compact. Since $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, $f(\bar{B})$ is also compact; thus bounded. The boundedness of $f(B)$ then follows from the fact that $f(B) \subseteq f(\bar{B})$.

2. No. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ and $K = [-1, 1]$. Then K is compact but $f^{-1}(K)$ is the whole real line so that $f^{-1}(K)$ is not compact.

3. No. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $C = [1, 4]$. Then C is connected since it is an interval (Corollary 3.69 in the lecture note) but $f^{-1}(C) = [-2, -1] \cup [1, 2]$ which is clearly disconnected. □

Problem 2. Consider a compact set $K \subseteq \mathbb{R}^n$ and let $f : K \to \mathbb{R}^m$ be continuous and one-to-one. Show that the inverse function $f^{-1} : f(K) \to K$ is continuous. How about if K is not compact but connected?

Proof. Let F be a closed subset of K. Then 1 of Problem 11 of Exercise 8 implies that F is compact. Therefore, $f(F)$ is compact since f is continuous (Theorem 4.25 in the lecture note). Since $f(F) = (f^{-1})^{-1}(F)$, we conclude that the pre-image of F under f^{-1} is compact; hence $(f^{-1})^{-1}(F)$ is closed in $f(K)$ for all closed sets $F \subseteq K$. Therefore, Theorem 4.14 in the lecture note shows that $f^{-1} : f(K) \to K$ is continuous.

However, $f^{-1} : f(K) \to K$ is not necessarily continuous if K is connected. For example, consider $f : [0, 2\pi) \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin t)$. Then f is one-to-one but $f^{-1} : f([0, 2\pi)) \to [0, 2\pi)$ is not continuous at $f(0) = (1, 0)$ since the sequences $\{\boldsymbol{x}_n\}_{n=1}^\infty, \{\boldsymbol{y}_n\}_{n=1}^\infty$ given by

$$\boldsymbol{x}_n = \Big(\cos\frac{1}{n}, \sin\frac{1}{n}\Big) \qquad \text{and} \qquad \boldsymbol{y}_n = \Big(\cos\big(2\pi - \frac{1}{n}\big), \sin\big(2\pi - \frac{1}{n}\big)\Big)$$

both converges to $(1, 0)$ but $f^{-1}(\boldsymbol{x}_n) = \dfrac{1}{n}$ and $f^{-1}(\boldsymbol{y}_n) = 2\pi - \dfrac{1}{n}$ so that

$$\lim_{n\to\infty} f^{-1}(\boldsymbol{x}_n) = 0 \neq 2\pi = \lim_{n\to\infty} f^{-1}(\boldsymbol{y}_n)\,.$$ □

Problem 3. Let (M, d) be a metric space, $K \subseteq M$ be compact, and $f : K \to \mathbb{R}$ be lower semi-continuous (see Problem 8 of Exercise 10 for the definition). Show that f attains its minimum on K.

Proof. **Claim**: there exists a sequence $\{x_n\}_{n=1}^\infty$ such that $\lim\limits_{n\to\infty} f(x_n) = \inf\limits_{x\in K} f(x)$.

Proof of claim: If $\inf\limits_{x\in K} f(x) \in \mathbb{R}$, for each $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$\inf_{x\in K} f(x) \leqslant f(x_n) \leqslant \inf_{x\in K} f(x) + \frac{1}{n}\,.$$

If $\inf\limits_{x\in K} f(x) = -\infty$, for each $n \in \mathbb{N}$ there exists $x_n \in K$ such that $f(x_n) < -n$. In either case, $\lim\limits_{n\to\infty} f(x_n) = \inf\limits_{x\in K} f(x)$. □

W.L.O.G. we can assume that $f(x_n) > \inf\limits_{x\in K} f(x)$ for all $n \in \mathbb{N}$ (for otherwise we find that f attains its minimum at some x_n). Let $n_1 = 1$, and for given n_k choose $n_{k+1} > n_k$ such that $f(x_{n_k}) > f(x_{n_{k+1}})$. In this way we obtain a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ satisfying that

$$\lim_{k\to\infty} f(x_{n_k}) = \inf_{x\in K} f(x) \qquad \text{and} \qquad f(x_{n_k}) \geqslant f(x_{n_{k+1}}) \quad \forall\, k \in \mathbb{N}\,.$$

Since $\{x_{n_k}\}_{k=1}^\infty \subseteq K$, by the compactness of K there exists a convergent subsequence $\big\{x_{n_{k_\ell}}\big\}_{\ell=1}^\infty$ of $\{x_{n_k}\}_{k=1}^\infty$. Suppose that $\lim\limits_{\ell\to\infty} x_{n_{k_\ell}} = a$. Then by the fact that $x_{n_k} \neq x_{n_\ell}$ for all $k \neq \ell$, we have

$$\#\big\{\ell \in \mathbb{N} \,\big|\, x_{n_{k_\ell}} = a\big\} \leqslant 1\,.$$

Therefore, up to deleting one term in the sequence we can assume that $\big\{x_{n_{k_\ell}}\big\}_{\ell=1}^\infty \subseteq K\backslash\{a\}$. In such a case the lower semi-continuity of f implies that

$$\liminf_{\ell\to\infty} f\big(x_{n_{k_\ell}}\big) \geqslant \liminf_{x\to a} f(x) \geqslant f(a)\,.$$

Since $\lim\limits_{n\to\infty} f(x_n) = \inf\limits_{x\in K} f(x)$, the inequality above implies that

$$\inf_{x\in K} f(x) = \liminf_{\ell\to\infty} f\big(x_{n_{k_\ell}}\big) \geqslant \liminf_{x\to a} f(x) \geqslant f(a) \geqslant \inf_{x\in K} f(x)\,;$$

thus $f(a) = \inf\limits_{x\in K} f(x)$. □

Problem 4. Let (M, d) be a metric space. Show that a subset $A \subseteq M$ is connected if and only if every continuous function defined on A whose range is a subset of $\{0, 1\}$ is constant.

Proof. "$\Rightarrow$" Assume that A is connected and $f : A \to \{0, 1\}$ is a continuous function, and $\delta = 1/2$. Suppose the contrary that $f^{-1}(\{0\}) \neq \varnothing$ and $f^{-1}(\{1\}) \neq \varnothing$. Then $A = f^{-1}((-\delta, \delta))$ and $B = f^{-1}((1-\delta, 1+\delta))$ are non-empty set. Moreover, the continuity of f implies that A and B are open relative to A; thus there exist open sets U and V such that

$$f^{-1}((-\delta, \delta)) = U \cap A \qquad \text{and} \qquad f^{-1}((1-\delta, 1+\delta)) = V \cap A\,.$$

Then

(1) $A \cap U \cap V = f^{-1}((-\delta, \delta)) \cap f^{-1}((1-\delta, 1+\delta)) = \varnothing\,,$

(2) $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing\,,$

(3) $A \subseteq U \cup V$ since the range of f is a subset of $\{0, 1\}\,;$

thus A is disconnect, a contradiction.

"$\Leftarrow$" Suppose the contrary that A is disconnected so that there exist open sets U and V such that

$$(1)\ A \cap U \cap V = \varnothing\,, \quad (2)\ A \cap U \neq \varnothing\,, \quad (3)\ A \cap V \neq \varnothing\,, \quad (4)\ A \subseteq U \cup V\,.$$

Let $f : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \cap U\,, \\ 1 & \text{if } x \in A \cap V\,. \end{cases}$$

We first prove that f is continuous on A. Let $a \in A$. Then $a \in A \cap U$ or $a \in A \cap V$. Suppose that $a \in A \cap U$. In particular $a \in U$; thus the openness of U provides $r > 0$ such that $B(a, r) \subseteq U$. Note that if $x \in B(a, r) \cap A$, then $x \in A \subseteq U$; thus

$$\big|f(x) - f(a)\big| = 0 \qquad \forall\, x \in B(a, r) \cap A$$

which shows the continuity of f at a. Similar argument can be applied to show that f is continuous at $a \in A \cap V$. □

Problem 5. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|$. Show that f is continuous on $(\mathbb{R}^n, \|\cdot\|_2)$.
Hint: Show that $|f(x) - f(y)| \leqslant C\|x - y\|_2$ for some fixed constant $C > 0$.

Problem 6. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector fields over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\dim(\mathcal{V}) < \infty$. Show that a subset K of $\mathcal{V}$ is compact if and only if K is closed and bounded.
Hint: See Remark 3.43 in the lecture note for the case $\|\cdot\| = \|\cdot\|_2$, and the general case follows from Example 4.29 in the lecture note.

In Exercise Problem 7 through 10, we focus on another kind of connected sets, so-called path connected sets. First we introduce path connectedness in the following

Definition 0.1. Let (M, d) be a metric space. A subset $A \subseteq M$ is said to be ***path connected*** if for every $x, y \in A$, there exists a continuous map $\varphi : [0, 1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

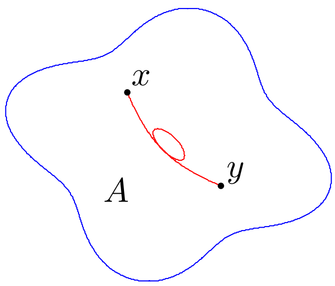

Figure 1: Path connected sets

Problem 7. Recall that a set A in a vector space $\mathcal{V}$ is called convex if for all $x, y \in A$, the line segment joining x and y lies in A. Show that a convex set in a normed space is path connected.

Problem 8. A set S in a vector space $\mathcal{V}$ is called ***star-shaped*** if there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. Show that a star-shaped set in a normed space is path connected.

Proof. Suppose that there exists $p \in S$ such that for any $q \in S$, the line segment joining p and q lies in S. In other words, such $p \in S$ satisfies that

$$(1-\lambda)q + \lambda p \subseteq S \qquad \forall\, \lambda \in [0,1] \text{ and } q \in S\,.$$

Let x, y in S. Define

$$\varphi(t) = \begin{cases} (1-2t)x + 2tp & \text{if } 0 \leqslant t \leqslant \dfrac{1}{2}\,, \\ (2-2t)p + (2t-1)y & \text{if } \dfrac{1}{2} < t \leqslant 1\,. \end{cases}$$

Then φ is continuous on $[0,1]$ $\big($since $\lim\limits_{t\to 0.5^+} \varphi(t) = \lim\limits_{t\to 0.5^-} \varphi(t) = p$ so that φ is continuous at $0.5\big)$. Moreover, $\varphi([0,0.5]) = \overline{xp}$ and $\varphi([0.5,1]) = \overline{py}$ so that $\varphi : [0,1] \to A$ is continuous with $\varphi(0) = x$ and $\varphi(1) = y$. Therefore, A is path connected. □

Problem 9. Let $A = \Big\{\big(x, \sin\frac{1}{x}\big) \,\Big|\, x \in (0,1]\Big\} \cup (\{0\} \times [-1,1])$. Show that A is connected in $(\mathbb{R}^2, \|\cdot\|_2)$, but A is not path connected.

Proof. Assume the contrary that A is path connected such that there is a continuous function $\varphi : [0,1] \to A$ such that $\varphi(0) = (x_0, y_0) \in \Big\{\big(x, \sin\frac{1}{x}\big) \,\big|\, x \in (0,1)\Big\}$ and $\varphi(1) = (0,0) \in \{0\} \times [-1,1]$. Let $t_0 = \inf\big\{t \in [0,1] \,\big|\, \varphi(t) \in \{0\} \times [-1,1]\big\}$. In other words, at $t = t_0$ the path touches $0 \times [-1,1]$ for the "first time". By the continuity of φ, $\varphi(t_0) \in \{0\} \times [-1,1]$. Since $\varphi(0) \notin \{0\} \times [-1,1]$, $\varphi([0,t_0)) \subseteq \Big\{\big(x, \sin\frac{1}{x}\big) \,\Big|\, x \in (0,1)\Big\}$.

Suppose that $\varphi(t_0) = (0, \bar{y})$ for some $\bar{y} \in [-1,1]$, and $\varphi(t) = \big(x(t), \sin\frac{1}{x(t)}\big)$ for $0 \leqslant t < t_0$. By the continuity of φ, there exists $\delta > 0$ such that if $|t - t_0| < \delta$, $|\varphi(t) - \varphi(t_0)| < 1$. In particular,

$$x(t)^2 + \Big(\sin\frac{1}{x(t)} - \bar{y}\Big)^2 < 1 \qquad \forall\, t \in (t_0 - \delta, t)\,.$$

On the other hand, since φ is continuous, $x(t)$ is continuous on $[0,t_0)$; thus by the fact that $[0,t_0)$ is connected, $x([0,t_0))$ is connected. Therefore, $x([0,t_0)) = (0,\bar{x}]$ for some $\bar{x} > 0$. Since $\lim\limits_{t\to t_0} x(t) = 0$, there exists $\{t_n\}_{n=1}^{\infty} \in [0,t_0)$ such that $t_n \to t_0$ as $n \to \infty$ and $\big|\sin\frac{1}{x(t_n)} - \bar{y}\big| \geqslant 1$. For $n \gg 1$, $t_n \in (t_0 - \delta, t_0)$ but

$$x(t_n)^2 + \Big(\sin\frac{1}{x(t_n)} - \bar{y}\Big)^2 \geqslant 1\,,$$

a contradiction.

On the other hand, A is the closure of the connected set $B = \Big\{\big(x, \sin\frac{1}{x}\big) \,\Big|\, x \in (0,1)\Big\}$ $\big($the connectedness of B follows from the fact that the function $\psi(x) = \big(x, \sin\frac{1}{x}\big)$ is continuous on the connected set $(0,1)\big)$. Therefore, by Problem 10 of Exercise 9, $A = \bar{B}$ is connected. □

Problem 10. Let (M, d) be a metric space, and $A \subseteq M$. Show that if A is path connected, then A is connected.

Proof. Assume the contrary that there are non-empty sets A_1, A_2 such that $A = A_1 \cup A_2$ but $A_1 \cap \overline{A_2} = A_2 \cap \overline{A_1} = \varnothing$. Let $x \in A_1$ and $y \in A_2$. By the path connectedness of A, there exists a continuous map $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Define $I_1 = \varphi^{-1}(A_1)$ and $I_2 = \varphi^{-1}(A_2)$. Then clearly $0 \in I_1$ and $1 \in I_2$, and $I_1 \cap I_2 = \varnothing$. Moreover,

$$[0,1] = \varphi^{-1}(A) = \varphi^{-1}(A_1 \cup A_2) = \varphi^{-1}(A_1) \cup \varphi^{-1}(A_2) = I_1 \cup I_2 \,.$$

Claim: $I_1 \cap \bar{I_2} = I_2 \cap \bar{I_1} = \varnothing$.

Suppose the contrary that $t \in I_1 \cap \bar{I_2}$. Then $t \in \varphi(A_1)$ which shows that $\varphi(t) \in A_1$. On the other hand, $t \in \bar{I_2}$; thus there exists $\{t_n\}_{n=1}^{\infty} \subseteq I_2$ such that $t_n \to t$ as $n \to \infty$. By the continuity of φ,

$$\varphi(t) = \lim_{n \to \infty} \varphi(t_n) \in \overline{A_2}\,;$$

thus we find that $\varphi(t) \in A_1 \cap \overline{A_2}$, a contradiction. Therefore, $I_1 \cap \bar{I_2} = \varnothing$. Similarly, $I_2 \cap \bar{I_1} = \varnothing$; thus we establish the existence of non-empty sets I_1 and I_2 such that

$$[0,1] = I_1 \cup I_2 \,, \quad I_1, I_2 \neq \varnothing \,, \quad I_1 \cap \bar{I_2} = I_2 \cap \bar{I_1} = \varnothing$$

which shows that $[0,1]$ is disconnected, a contradiction. □

Alternative proof. Assume the contrary that there are two open sets V_1 and V_2 such that

1. $A \cap V_1 \cap V_2 = \varnothing$; 2. $A \cap V_1 \neq \varnothing$; 3. $A \cap V_2 \neq \varnothing$; 4. $A \subseteq V_1 \cup V_2$.

Since A is path connected, for $x \in A \cap V_1$ and $y \in A \cap V_2$, there exists a continuous map $\varphi : [0,1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. By Theorem 4.14 in the lecture note, there exist U_1 and U_2 open in $(\mathbb{R}, |\cdot|)$ such that $\varphi^{-1}(V_1) = U_1 \cap [0,1]$ and $\varphi^{-1}(V_2) = U_2 \cap [0,1]$. Therefore,

$$[0,1] = \varphi^{-1}(A) \subseteq \varphi^{-1}(V_1) \cup \varphi^{-1}(V_2) \subseteq U_1 \cup U_2 \,.$$

Since $0 \in U_1$, $1 \in U_2$, and $[0,1] \cap U_1 \cap U_2 = \varphi^{-1}(A \cap V_1 \cap V_2) = \varnothing$, we conclude that $[0,1]$ is disconnected, a contradiction to Theorem 3.68 in the lecture note. □

Problem 11. Let (M, d), (N, ρ) be metric spaces, A be a subset of M, and $f : A \to N$ be a continuous map. Show that if $C \subseteq A$ is path connected, so is $f(C)$.

Proof. Let $y_1, y_2 \in f(C)$. Then $\exists\, x_1, x_2 \in C$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since C is path connected, $\exists\, r : [0,1] \to C$ such that r is continuous on $[0,1]$ and $r(0) = x_1$ and $r(1) = x_2$. Let $\varphi : [0,1] \to f(C)$ be defined by $\varphi = f \circ r$. By Corollary 4.24 in the lecture note φ is continuous on $[0,1]$, and $\varphi(0) = y_1$ and $\varphi(1) = y_2$. □