Exercise Problem Sets 10

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Problem 1 (True or False). Determine whether the following statements are true or false. If it is true, prove it. Otherwise, give a counter-example.

1. Every open set in a metric space is a countable union of closed sets.
2. Let $A \subseteq \mathbb{R}$ be bounded from above, then $\sup A \in A'$.
3. An infinite union of distinct closed sets cannot be closed.
4. An interior point of a subset A of a metric space (M, d) is an accumulation point of that set.
5. Let (M, d) be a metric space, and $A \subseteq M$. Then $(A')' = A'$.
6. There exists a metric space in which some unbounded Cauchy sequence exists.
7. Every metric defined in $\mathbb{R}^n$ is induced from some "norm" in $\mathbb{R}^n$.
8. There exists a non-zero dimensional normed vector space in which some compact non-zero dimensional linear subspace exists.
9. There exists a set $A \subseteq (0, 1]$ which is compact in $(0, 1]$ (in the sense of subspace topology), but A is not compact in $\mathbb{R}$.
10. Let $A \subseteq \mathbb{R}^n$ be a non-empty set. Then a subset B of A is compact in A if and only if B is closed and bounded in A.

Solution. 1. **True**. We note that the statement above is equivalent to that "every closed set in a metric space is a countable intersection of open sets". To see that this equivalent statement is true, we let F be a closed set. For each $n \in \mathbb{N}$, define

$$U_n = \bigcup_{x \in F} B\big(x, \frac{1}{n}\big).$$

Then $F \subseteq U_n$ $\big($since each point $x \in F$ belongs to the ball $B\big(x, \frac{1}{n}\big)\big)$. Moreover, U_n is open since it is the union of open sets.

Claim: $F = \bigcap_{n=1}^{\infty} U_n$.

Proof of claim: Since $F \subseteq U_n$ for all $n \in \mathbb{N}$, $F \subseteq \bigcap_{n=1}^{\infty} U_n$; thus it suffices to shows that $F \supseteq \bigcap_{n=1}^{\infty} U_n$ or equivalently, $F^\complement \subseteq \bigcup_{n=1}^{\infty} U_n^\complement$. To see the inclusion, we let $x \in F^\complement$ and use the closedness of F to find an $n_0 \in \mathbb{N}$ such that $B(x, \frac{1}{n_0}) \subseteq F^\complement$. This implies that $d(x, y) \geqslant \frac{1}{n_0}$ for all $y \in F$; thus $x \notin U_{n_0}$. Therefore, $x \in U_{n_0}^\complement$ so that $x \in \bigcup_{n=1}^{\infty} U_n^\complement$. □

2. **False**. Let A be a collection of single point $\{a\}$. Then A is bounded from above and $\sup A = a$ but $A' = \varnothing$.

3. **False**. Consider the union of the family of closed sets $\big\{[3n-1, 3n+1] \,\big|\, n \in \mathbb{N}\big\}$. We note that for $n \neq m$ the two sets $[3n-1, 3n+1] \cap [3m-1, 3m+1] = \varnothing$ so that this family is a collection of distinct set and $\bigcup\limits_{n=1}^{\infty} [3n-1, 3n+1]$ is closed.

4. **False**. Every point x in a discrete metric is the only point in the set $B(x,1)$ so that $x \notin B(x,1)'$.

5. **False**. A counter-example can be found in 5 of Problem 3 in Exercise 8.

6. **False**. By Proposition 2.58 in the lecture note, every Cauchy sequence is bounded.

7. **False**. The discrete metric d_0 on $\mathbb{R}^n$ cannot be induced by a norm since every set in $(\mathbb{R}^n, d_0)$ is bounded but $\mathbb{R}^n$ is unbounded in $(\mathbb{R}^n, \|\cdot\|)$ for any norms $\|\cdot\|$ on $\mathbb{R}^n$.

8. **False**. Note that any non-zero dimensional linear subspace of a normed space is unbounded; thus any non-zero dimensional linear subspace cannot be compact since a compact set must be bounded.

9. **False**. By Theorem 3.77 in the lecture note, A is compact in $(0,1]$ if and only if A is compact in $\mathbb{R}$.

10. **False**. By Theorem 3.42 in the lecture note, it is true that B is compact in A then B is closed and bounded in A; however, the reverse statement if not true. For example, if $A = B = (0,1)$, then B is closed and bounded in A but B is not compact in $\mathbb{R}$. □

Problem 2. Let (M,d) be a metric space, and $A \subseteq M$ be a subset. Determine which of the following statements are true.

1. $\text{int}A = A \backslash \partial A$.

2. $\text{cl}(A) = M \backslash \text{int}(M \backslash A)$.

3. $\text{int}(\text{cl}(A)) = \text{int}(A)$.

4. $\text{cl}(\text{int}(A)) = A$.

5. $\partial(\text{cl}(A)) = \partial A$.

6. If A is open, then $\partial A \subseteq M \backslash A$.

7. If A is open, then $A = \text{cl}(A) \backslash \partial A$. How about if A is not open?

Solution. 1. **True**. First we note that $\mathring{A} \subseteq A$ and $\mathring{A} \cap \partial A = \varnothing$. Therefore,

$$\mathring{A} \subseteq A \backslash \partial A\,.$$

On the other hand, if $x \in A\backslash\partial A$, by the fact that $\partial A = \bar{A} \cap \overline{A^\complement}$, we find that x is not a limit point of $A^\complement$; thus there exists $r > 0$ such that $B(x,r) \subseteq (A^\complement)^\complement = A$. This Remark 3.3 in the lecture note implies that $x \in \mathring{A}$ so that $A\backslash\partial A \subseteq \mathring{A}$.

2. **True**. Note that $x \notin \mathring{B}$ if and only if there exists $\{x_n\}_{n=1}^\infty \subseteq B^\complement$ such that $\lim\limits_{n\to\infty} x_n = x$. Therefore,

$$\begin{aligned} x \in \bar{A} &\Leftrightarrow \big(\exists\,\{x_n\}_{n=1}^\infty \subseteq A\big)\big(\lim_{n\to\infty} x_n = x\big) \Leftrightarrow \big(\exists\,\{x_n\}_{n=1}^\infty \subseteq (M\backslash A)^\complement\big)\big(\lim_{n\to\infty} x_n = x\big) \\ &\Leftrightarrow x \notin \mathrm{int}(M\backslash A) \Leftrightarrow x \in M\backslash \mathrm{int}(M\backslash A)\,. \end{aligned}$$

3. **False**. Let $A = [0,1] \cap \mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$. Then $\mathrm{cl}(A) = [0,1]$ and $\mathrm{int}(A) = \varnothing$ so that $\mathrm{int}(\mathrm{cl}(A)) = (0,1) \neq \mathrm{int}(A)$.

4. **False**. Let $A = [0,1] \cap \mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$. Then $\mathrm{int}(A) = \varnothing$ so that $\mathrm{cl}(\mathrm{int}(A)) = \varnothing \neq A$.

5. **False**. Let $A = [0,1] \cap \mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$. Then $\bar{A} = [0,1]$ so that $\partial\bar{A} = \{0,1\} \neq \partial A$.

6. **True**. If A is open, then every point $x \in A$ is an interior point so that $x \notin \partial A$ (if $x \in \partial A$, then there exists $\{x_n\}_{n=1}^\infty \subseteq A^\complement$ such that $\lim\limits_{n\to\infty} x_n = x$ so that $x \notin \mathring{A}$).

7. **True**. By Proposition 3.13 in the lecture note, $\partial A = \bar{A}\backslash\mathring{A}$; thus the fact that $\mathring{A} \subseteq \bar{A}$ shows that $\bar{A} = \mathring{A} \cup \partial A$. Since $\partial A \cap \mathring{A} = \varnothing$, we find that $A = \bar{A}\backslash\partial A$.

   If A is not open, the statement is false. For example, consider $A = [0,1]$ in $(\mathbb{R}, |\cdot|)$. Then A is not open and $\bar{A} = [0,1]$ and $\partial A = \{0,1\}$ so that $\bar{A}\backslash\partial A = (0,1) \neq A$. □

Problem 3. Complete the following.

1. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \quad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

   exist but are not equal.

2. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the two limits above exist and are equal but f is not continuous.

3. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through the origin but is not continuous.

Problem 4. Complete the following.

1. Show that the projection map $f : \begin{array}{rcl} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x \end{array}$ is continuous.

2. Show that if $U \subseteq \mathbb{R}$ is open, then $A = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, x \in U\big\}$ is open.

3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $U \subseteq \mathbb{R}$ such that $f(U)$ is not open.

Problem 5. Show that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, is continuous if and only if for every $B \subseteq A$,

$$f(\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B)).$$

Proof. "$\Rightarrow$" Let $B \subseteq A$ and $y \in f(\mathrm{cl}(B) \cap A)$. Then there exists $x \in \mathrm{cl}(B) \cap A$ such that $y = f(x)$. By the property of $\bar{B}$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq B$ such that $\lim_{n\to\infty} x_n = x$. Since $B \subseteq A$, $\{x_n\}_{n=1}^{\infty} \subseteq A$; thus the continuity of f (at x) implies that

$$\lim_{n\to\infty} f(x_n) = f(x).$$

On the other hand, $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence in $f(B)$, so the limit $f(x)$ must belong to $\mathrm{cl}(f(B))$. Therefore, $y = f(x) \in \mathrm{cl}(f(B))$ which shows the inclusion $f((\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B))$.

"$\Leftarrow$" Suppose the contrary that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ with limit $x \in A \cap A'$ such that $\lim_{n\to\infty} f(x_n) \neq f(x)$. Then there exists $\varepsilon > 0$ such that for all $N > 0$ there exists $n \geqslant N$ such that $\|f(x_n) - f(x)\| \geqslant \varepsilon$. Let $n_1 \in \mathbb{N}$ be such that $\|f(x_{n_1} - f(x)\| \geqslant \varepsilon$. Let $n_2 > n_1$ such that $\|f(x_{n_2}) - f(x)\| \geqslant \varepsilon$. Continuing this process, we obtain an increasing sequence $\{n_j\}_{j=1}^{\infty}$ such that

$$\|f(x_{n_j}) - f(x)\| \geqslant \varepsilon \qquad \forall\, j \in \mathbb{N}. \tag{0.1}$$

Let $B = \{x_{n_j}\}$. Then $x \in \bar{B}$ since $\lim_{n\to\infty} x_n = x$ $\big($so that $\lim_{j\to\infty} x_{n_j} = x\big)$. On the other hand, (0.1) implies that $f(x) \notin \mathrm{cl}(f(B))$ since $B(f(x), \varepsilon) \cap f(B) = \varnothing$. Therefore,

$$f(\mathrm{cl}(B) \cap A) \nsubseteq \mathrm{cl}(f(B)),$$

a contradiction. □

Problem 6. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfy $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^n$.

1. Show that $T(rx) = rT(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}^n$.

2. Suppose that T is continuous on $\mathbb{R}^n$. Show that T is linear; that is, $T(cx + y) = cT(x) + T(y)$ for all $c \in \mathbb{R}$, $x, y \in \mathbb{R}^n$.

3. Suppose that T is continuous at some point x_0 in $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

4. Suppose that T is bounded on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

5. Suppose that T is bounded from above (or below) on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

6. Construct a $T : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$, but $T(x+y) = T(x)+T(y)$ for all $x, y \in \mathbb{R}$.

Proof. 1. By induction, $T(kx) = kT(x)$ for all $k \in \mathbb{N}$. Moreover, $T(0) = T(0+0) = T(0) + T(0)$ which implies that $T(0) = 0$; thus $T(0x) = 0T(x)$ and if $k \in \mathbb{N}$,

$$-kT(x) = -kT(x)+T(0) = -kT(x)+T(kx+(-kx)) = -kT(x)+T(kx)+T(-kx) = T(-kx)\,.$$

Therefore, $T(kx) = kT(x)$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Let $r = \dfrac{q}{p}$ for some $p, q \in \mathbb{Z}$. Then for $x \in \mathbb{R}^n$,

$$pT(rx) = T(prx) = T(qx) = qT(x)$$

which implies that $T(rx) = rT(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}^n$.

2. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then there exists $\{c_k\}_{k=1}^{\infty} \subseteq \mathbb{Q}$ such that $\lim\limits_{k\to\infty} c_k = c$. This further implies that $c_k x \to cx$ as $k \to \infty$ since

$$\lim_{k\to\infty} \|c_n x - cx\| = \lim_{k\to\infty} \|(c_k - c)x\| = \|x\| \lim_{k\to\infty} |c_k - c| = 0$$

Therefore, by the continuity of T,

$$T(cx+y) = T(cx) + T(y) = \lim_{k\to\infty} T(c_k x) + T(y) = \lim_{k\to\infty} c_k T(x) + T(y) = cT(x) + T(y)\,.$$

3. Let $a \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. By the continuity of T at x_0, there exists $\delta > 0$ such that

$$\|T(x-x_0)\| = \|T(x) - T(x_0)\| < \varepsilon \quad \text{whenever} \quad \|x - x_0\| < \delta\,.$$

The statement above implies that if $\|x\| < \delta$, then $\|T(x)\| < \varepsilon$. Therefore,

$$\|T(x) - T(a)\| = \|T(x-a)\| < \varepsilon \quad \text{whenever} \quad \|x - a\| < \delta$$

which shows that T is continuous at a.

4. Suppose that T is bounded on an open set U so that $T(U) \subseteq B(0, M)$. Let $x_0 \in U$. Then there exists $r > 0$ such that $B(x_0, r) \subseteq U$. Therefore, if $x \in B(0, r)$, then $x + x_0 \in B(x_0, r)$ so that

$$\|T(x)\| \leqslant \|T(x+x_0)\| + \|T(x_0)\| \leqslant M + \|T(x_0)\| \equiv R\,;$$

thus we establish that there exists r and R such that

$$\|T(x)\| \leqslant R \quad \text{whenever} \quad \|x\| < r\,.$$

Let $\varepsilon > 0$ be given. Choose $c \in \mathbb{Q}$ so that $0 < c < \dfrac{\varepsilon}{R}$. For such a fixed $c \in \mathbb{Q}$, choose $0 < \delta < cr$. If $\|x\| < \delta$, then $\big\|\dfrac{x}{c}\big\| < \dfrac{\delta}{c} < r$; thus if $\|x\| < \delta$, we have $\big\|T\big(\dfrac{x}{c}\big)\big\| \leqslant R$ so that

$$\|T(x)\| = \Big\|T\Big(c\frac{x}{c}\Big)\Big\| = \Big\|cT\Big(\frac{x}{c}\Big)\Big\| = c\Big\|T\Big(\frac{x}{c}\Big)\Big\| \leqslant cR < \varepsilon\,.$$

Therefore, T is continuous at 0. By 3, T is continuous on $\mathbb{R}^n$.

5. Suppose that $Tx \leqslant M$ (so that in this case $m = 1$) for all $x \in U$, where U is an open set in $\mathbb{R}^n$. Let $x_0 \in U$. Then there exists $r > 0$ such that $B(x_0, r) \subseteq U$; thus if $x \in B(0, r)$,

$$Tx = T(x + x_0) - T(x_0) \leqslant M - T(x_0) \equiv R .$$

Therefore, we establish that there exist r and R such that

$$T(x) \leqslant R \quad \text{ whenever } \quad x \in B(0, r) .$$

For $x \in B(0, r)$, we must have $-x \in B(0, r)$; thus

$$-T(x) = T(-x) \leqslant R ;$$

thus $-R \leqslant T(x)$ whenever $x \in B(0, r)$. Therefore, $|T(x)| \leqslant R$ whenever $\|x\| < r$. By 4, T is continuous on $\mathbb{R}^n$. □

Problem 7. Let (M, d) be a metric space, $A \subseteq M$, and $f : A \to \mathbb{R}$. For $a \in A'$, define

$$\liminf_{x \to a} f(x) = \lim_{r \to 0^+} \inf \{ f(x) \,|\, x \in B(a, r) \cap A \backslash \{a\} \} ,$$
$$\limsup_{x \to a} f(x) = \lim_{r \to 0^+} \sup \{ f(x) \,|\, x \in B(a, r) \cap A \backslash \{a\} \} .$$

Complete the following.

1. Show that both $\liminf\limits_{x \to a} f(x)$ and $\limsup\limits_{x \to a} f(x)$ exist (which may be $\pm\infty$), and

$$\liminf_{x \to a} f(x) \leqslant \limsup_{x \to a} f(x) .$$

Furthermore, there exist sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \subseteq A \backslash \{a\}$ such that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both converge to a, and

$$\lim_{n \to \infty} f(x_n) = \liminf_{x \to a} f(x) \qquad \text{and} \qquad \lim_{n \to \infty} f(y_n) = \limsup_{x \to a} f(x) .$$

2. Let $\{x_n\}_{n=1}^{\infty} \subseteq A \backslash \{a\}$ be a convergent sequence with limit a. Show that

$$\liminf_{x \to a} f(x) \leqslant \liminf_{n \to \infty} f(x_n) \leqslant \limsup_{n \to \infty} f(y_n) \leqslant \limsup_{x \to a} f(x) .$$

3. Show that $\lim\limits_{x \to a} f(x) = \ell$ if and only if

$$\liminf_{x \to a} f(x) = \limsup_{x \to a} f(x) = \ell .$$

4. Show that $\liminf\limits_{x \to a} f(x) = \ell \in \mathbb{R}$ if and only if the following two conditions hold:

   (a) for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\ell - \varepsilon < f(x)$ for all $x \in B(a, \delta) \cap A \backslash \{a\}$;

   (b) for all $\varepsilon > 0$ and $\delta > 0$, there exists $x \in B(a, \delta) \cap A \backslash \{a\}$ such that $f(x) < \ell + \varepsilon$.

Formulate a similar criterion for limsup and for the case that $\ell = \pm\infty$.

5. Compute the liminf and limsup of the following functions at any point of $\mathbb{R}$.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^\complement\,, \\ \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \text{ with } (p,q) = 1,\ q > 0,\ p \neq 0\,. \end{cases}$

   (b) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}\,, \\ -x & \text{if } x \in \mathbb{Q}^\complement\,. \end{cases}$

Proof. For $r > 0$, define $m, M : A' \to R^*$ by

$$m(r) = \inf\big\{f(x)\,\big|\, x \in B(a,r) \cap A\backslash\{a\}\big\} \quad \text{and} \quad M(r) = \sup\big\{f(x)\,\big|\, x \in B(a,r) \cap A\backslash\{a\}\big\}\,.$$

We remark that it is possible that $m(r) = -\infty$ or $M(r) = \infty$. Note that m is decreasing and M is increasing in $(0, \infty)$.

1. By the monotonicity of m and M, $\lim\limits_{r\to0^+} m(r)$ and $\lim\limits_{r\to0^+} M(r)$ "exist" (which may be $\pm\infty$). Moreover, $m(r) \leqslant M(r)$ for all $r > 0$; thus $\lim\limits_{r\to0^+} m(r) \leqslant \lim\limits_{r\to0^+} M(r)$ so we conclude that

$$\liminf_{x\to a} f(x) = \lim_{r\to0^+} m(r) \leqslant \lim_{r\to0^+} M(r) = \limsup_{x\to a} f(x)\,.$$

   Since $\liminf\limits_{x\to a} f(x) = -\limsup\limits_{x\to a}(-f)(x)$, it suffices to consider the case of the limit superior.

   (a) If $\limsup\limits_{x\to a} f(x) = \infty$, then for each $n \in \mathbb{N}$ there exists $0 < \delta_n < \dfrac{1}{n}$ such that

$$M(r) \geqslant n \quad \textit{whenever} \quad 0 < r < \delta_n\,.$$

   By the definition of the supremum, for each $n \in \mathbb{N}$ there exists $x_n \in B\big(a, \frac{\delta_n}{2}\big) \cap A\backslash\{a\}$ such that $f(x_n) \geqslant n - 1$.

   (b) If $\limsup\limits_{x\to a} f(x) = L$, then for each $n \in \mathbb{N}$ there exists $0 < \delta_n < \dfrac{1}{n}$ such that

$$\big|M(r) - L\big| < \frac{1}{n} \quad \textit{whenever} \quad 0 < r < \delta_n\,.$$

   By the definition of the supremum, for each $n \in \mathbb{N}$ there exists $x_n \in B\big(a, \frac{\delta_n}{2}\big) \cap A\backslash\{a\}$ such that

$$L - \frac{1}{n} < f(x_n) < L + \frac{1}{n}\,.$$

   Since $\delta_n \to 0$ as $n \to \infty$, we find that $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ converges to a and $\lim\limits_{n\to\infty} f(x_n) = \limsup\limits_{x\to a} f(x)$.

2. It suffices to show the case of the limit inferior. Let $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ and $x_n \to a$ as $n \to \infty$. For every $k \in \mathbb{N}$, there exists $N_k > 0$ such that $0 < d(x_n, a) < \dfrac{1}{k}$ whenever $n \geqslant N_k$. W.L.O.G.,

we can assume that $N_k \geqslant k$ and $N_{k+1} > N_k$ for all $k \in \mathbb{N}$. By the definition of infimum,

$$m\big(\frac{1}{k}\big) \leqslant f(x_n) \quad \textit{whenever} \quad n \geqslant N_k$$

which further implies that

$$m\big(\frac{1}{k}\big) \leqslant \inf_{n \geqslant N_k} f(x_n) \,.$$

Note that $\lim\limits_{r \to 0^+} m(r) = \lim\limits_{k \to \infty} m\big(\frac{1}{k}\big)$ and $\lim\limits_{k \to \infty} \inf\limits_{n \geqslant N_k} f(x_n) = \lim\limits_{k \to \infty} \inf\limits_{n \geqslant k} f(x_n)$ (the latter follows from the fact that $\big\{ \inf\limits_{n \geqslant N_k} f(x_n) \big\}_{k=1}^{\infty}$ is a subsequence of the "convergent" sequence $\big\{ \inf\limits_{n \geqslant k} f(x_n) \big\}_{k=1}^{\infty}$), we conclude that

$$\liminf_{x \to a} f(x) = \lim_{r \to 0^+} m(r) = \lim_{k \to \infty} m\big(\frac{1}{k}\big) \leqslant \lim_{k \to \infty} \inf_{n \geqslant N_k} f(x_n) = \lim_{k \to \infty} \inf_{n \geqslant k} f(x_n) = \liminf_{n \to \infty} f(x_n) \,.$$

3. ($\Rightarrow$) Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \qquad \text{whenever} \quad x \in B(a, \delta) \cap A \backslash \{a\} \,.$$

Therefore,

$$\ell - \varepsilon < f(x) < \ell + \varepsilon \qquad \text{whenever} \quad x \in B(a, \delta) \cap A \backslash \{a\}$$

which implies that

$$\ell - \varepsilon \leqslant m(\delta) \leqslant M(\delta) \leqslant \ell + \varepsilon \,.$$

By the monotonicity of m and M, the inequality above implies that

$$\ell - \varepsilon \leqslant m(\delta) \leqslant m(r) \leqslant M(r) \leqslant M(\delta) \leqslant \ell + \varepsilon \quad \forall\, 0 < r < \delta \,.$$

Passing to the limit as $r \to 0^+$, we find that

$$\ell - \varepsilon \leqslant \liminf_{x \to a} f(x) \leqslant \limsup_{x \to a} f(x) \leqslant \ell + \varepsilon \,.$$

Since $\varepsilon > 0$ is chosen arbitrary, we conclude that $\liminf\limits_{x \to a} f(x) = \limsup\limits_{x \to a} f(x) = \ell$.

($\Leftarrow$) Let $\{x_n\}_{n=1}^{\infty} \subseteq A \backslash \{a\}$ be a sequence with limit a. Then 2 and the assumption that $\liminf\limits_{x \to a} f(x) = \limsup\limits_{x \to a} f(x) = \ell$ imply that $\liminf\limits_{n \to \infty} f(x_n) = \limsup\limits_{n \to \infty} f(x_n) = \ell$. Therefore, $\lim\limits_{n \to \infty} f(x_n) = \ell$.

4. ($\Rightarrow$) This direction is proved by contradiction.

   (a) Suppose the contrary that there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $x_n \in B\big(a, \frac{1}{n}\big) \cap A \backslash \{a\}$ such that $f(x_n) \leqslant \ell - \varepsilon$. Then $\{x_n\}_{n=1}^{\infty} A \backslash \{a\}$ and $\lim\limits_{n \to \infty} x_n = a$; however,

$$\liminf_{n \to \infty} f(x_n) \leqslant \ell - \varepsilon < \ell = \liminf_{x \to a} f(x) \,,$$

   a contradiction to 2.

(b) Suppose the contrary that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$f(x) \geqslant \ell + \varepsilon \qquad \forall\, x \in B(a,\delta) \cap A\backslash\{a\}\,.$$

Then $m(\delta) \geqslant \ell + \varepsilon$; thus the monotonicity of m implies that

$$\ell + \varepsilon \leqslant m(\delta) \leqslant m(r) \quad \text{whenever} \quad 0 < r < \delta\,.$$

Passing to the limit as $r \to 0^+$, we conclude that

$$\ell + \varepsilon \leqslant \lim_{r\to 0^+} m(r) = \liminf_{x\to a} f(x)\,,$$

a contradiction.

($\Leftarrow$) Let $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ be a sequence with limit a, and $\varepsilon > 0$ be given. Then (a) provides $\delta > 0$ such that $f(x) > \ell - \varepsilon$ whenever $x \in B(a,\delta) \cap A\backslash\{a\}$. For such $\delta > 0$, there exists $N > 0$ such that $0 < d(x_n, a) < \delta$ for all $n \geqslant N$. Therefore, if $n \geqslant N$, $f(x_n) > \ell - \varepsilon$ which implies that $\liminf\limits_{n\to\infty} f(x_n) \geqslant \ell - \varepsilon$. Since $\varepsilon > 0$ is chosen arbitrary, we conclude that

$$\liminf_{n\to\infty} f(x_n) \geqslant \ell \text{ for every convergent sequence } \{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\} \text{ with limit } a.$$

On the other hand, using (b) we find that for each $n \in \mathbb{N}$, there exists $x_n \in B\big(a, \frac{1}{n}\big) \cap A\backslash\{a\}$ such that $f(x_n) < \ell + \frac{1}{n}$. Then $\liminf\limits_{n\to\infty} f(x_n) \leqslant \ell$, and (i) further implies that $\liminf\limits_{n\to\infty} f(x_n) = \ell$; thus we establish that there exists a convergent sequence $\{x_n\}_{n=1}^\infty \subseteq A\backslash\{a\}$ with limit a such that $\liminf\limits_{n\to\infty} f(x_n) = \ell$.

By 1 and 2, we conclude that $\ell = \liminf\limits_{x\to a} f(x)$.

5. (a) $\liminf\limits_{x\to a} f(x) = \limsup\limits_{x\to a} f(x) = 0$ for all $a \in \mathbb{R}$.

   (b) $\liminf\limits_{x\to a} f(x) = -|a|$, $\limsup\limits_{x\to a} f(x) = |a|$. In particular, $\lim\limits_{x\to 0} f(x) = 0$. □

Problem 8. Let (M,d) be a metric space, and $A \subseteq M$. A function $f : A \to \mathbb{R}$ is called ***lower semi-continuous*** / ***upper semi-continuous*** at $a \in A$ if either $a \in A\backslash A'$ or $\liminf\limits_{x\to a} f(x) \geqslant f(a)$, / $\limsup\limits_{x\to a} f(x) \leqslant f(a)$, and is called lower/upper semi-continuous on A if f is lower/uppser semi-continuous at a for all $a \in A$.

1. Show that $f : A \to \mathbb{R}$ is lower semi-continuous on A if and only if $f^{-1}((-\infty, r])$ is closed relative to A. Also show that $f : A \to \mathbb{R}$ is upper semi-continuous on A if and only if $f^{-1}([r,\infty))$ is closed relative to A.

2. Show that f is lower semi-continuous on A if and only if for all convergent sequences $\{x_n\}_{n=1}^\infty \subseteq A$ and $\{s_n\}_{n=1}^\infty \subseteq \mathbb{R}$ satisfying $f(x_n) \leqslant s_n$ for all $n \in \mathbb{N}$, we have

$$f\big(\lim_{n\to\infty} x_n\big) \leqslant \lim_{n\to\infty} s_n\,.$$

3. Let $\{f_\alpha\}_{\alpha\in I}$ be a family of lower semi-continuous functions on A. Prove that $f(x) = \sup\limits_{\alpha\in I} f_\alpha(x)$ is lower semi-continuous on A.

4. Let A be a perfect set (that is, A contains no isolated points) and $f : A \to \mathbb{R}$ be given. Define
$$f^*(x) = \limsup_{y\to x} f(y) \qquad \text{and} \qquad f_*(x) = \liminf_{y\to x} f(y)\,.$$
Show that f^* is upper semi-continuous and f_* is lower semi-continuous, and $f_*(x) \leqslant f(x) \leqslant f^*(x)$ for all $x \in A$. Moreover, if g is a lower semi-continuous function on A such that $g(x) \leqslant f(x)$ for all $x \in A$, then $g \leqslant f_*$.

Proof. We first note that by 1, 2 and 4 of Problem 7,
$$\begin{aligned} &f : A \to \mathbb{R} \text{ is lower semi-continuous at } a \\ &\Leftrightarrow (\forall\, \varepsilon > 0)(\exists\, \delta > 0)\big(x \in B(a,\delta)\cap A \Rightarrow f(x) > f(a) - \varepsilon\big) \\ &\Leftrightarrow \big(\forall\, \{x_n\}_{n=1}^\infty \subseteq A\big)\Big(\lim_{n\to\infty} x_n = a \Rightarrow \liminf_{n\to\infty} f(x_n) \geqslant f(a)\Big)\,. \end{aligned} \tag{0.2}$$
We note that the first statement implies the second one because of 4(a) in Problem 7, the second statement implies the third one because of $x_n \in B(a,\delta)\cap A$ when $n \gg 1$, and the third statement implies the first one because of 1 in Problem 7.

1. ($\Rightarrow$) It suffices to prove the case for limit inferior since $\limsup\limits_{x\to a} f(x) = -\liminf\limits_{x\to a}(-f)(x)$. We note that E is closed relative to A if and only if $E \cap A$ is a closed set in the metric space (A, d). Therefore, a subset of E of A is closed relative to A if and only if every sequence $\{x_n\}_{n=1}^\infty \subseteq E$ that converges to a point in A must also has limit in E.
   Let $r \in \mathbb{R}$ and $\{x_n\}_{n=1}^\infty$ be a sequence in $E \equiv f^{-1}((-\infty, r])$ such that $\{x_n\}_{n=1}^\infty$ converges to a point $a \in A$. Then $f(a) \leqslant \liminf\limits_{n\to\infty} f(x_n) \leqslant r$ which implies that $a \in f^{-1}((-\infty, r])$.
   ($\Leftarrow$) Let $a \in A$ and $\varepsilon > 0$ be given. Define $r = f(a) - \varepsilon$. Then $V = f^{-1}((r,\infty))$ is open relative to A (since $f^{-1}((-\infty, r])$ is closed relative to A). Since $a \in V$, there exists $\delta > 0$ such that $B(a,\delta)\cap A \subseteq V$. This implies that
$$f(a) - \varepsilon < f(x) \qquad \forall\, x \in B(a,\delta)\cap A\,.$$
   Therefore, the equivalence (0.2) shows that f is lower semi-continuous at a.

2. ($\Rightarrow$) Let $\{x_n\}_{n=1}^\infty$ be a convergent sequence in A with limit a, $\{s_n\}_{n=1}^\infty$ be a real sequence with limit s, and $f(x_n) \leqslant s_n$ for all $n \in \mathbb{N}$. Suppose that $f(a) > s$. Let $\varepsilon = \dfrac{f(a) - s}{2}$. Since f is lower semi-continuous at a, $\liminf\limits_{x\to a} f(x) \geqslant f(a)$; thus there exists $\delta > 0$ such that
$$f(a) - \varepsilon < f(x) \qquad \forall\, x \in B(a,\delta)\cap A\,.$$
   On the other hand, there exists $N > 0$ such that $x_n \in B(a,\delta)\cap A$ and $s_n < s+\varepsilon$ whenever $n \geqslant N$. Therefore, if $n \geqslant N$,
$$s_n < s + \varepsilon = f(a) - \varepsilon < f(x_n)\,,$$
   a contradiction.

($\Leftarrow$) Let $a \in A$, and $\{x_n\}_{n=1}^{\infty} \subseteq A$ be a sequence with limit a. Let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $\lim\limits_{j\to\infty} f(x_{n_j}) = \liminf\limits_{n\to\infty} f(x_n)$. Define $s_j = f(x_{n_j})$. Then clearly $f(x_{n_j}) \leqslant s_j$ for all $j \in \mathbb{N}$; thus by assumption

$$f(a) \leqslant \lim_{j\to\infty} s_j = \liminf_{n\to\infty} f(x_n)\,.$$

3. Let $a \in A \cap A'$ and $\{x_n\}_{n=1}^{\infty} \subseteq A\backslash\{a\}$ be a sequence with limit a. Then $f_\alpha(x_n) \leqslant f(x_n)$ for all $n \in \mathbb{N}$ and $\alpha \in I$. Since f_α is lower semi-continuous for each $\alpha \in I$, for $\alpha \in I$ we have

$$f_\alpha(a) \leqslant \liminf_{x\to a} f_\alpha(x) \leqslant \liminf_{x\to a} f(x)\,.$$

The inequality above implies that

$$f(a) = \sup_{\alpha\in I} f_\alpha(a) \leqslant \liminf_{x\to a} f(x)\,;$$

thus f is lower semi-continuous at a. ◻