Exercise Problem Sets 4

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Problem 1. Let $b \in \mathbb{R}$ and $b > 1$.

1. Show the law of exponents holds (for rational exponents); that is, show that

   (a) if r, s in $\mathbb{Q}$, then $b^{r+s} = b^r \cdot b^s$.

   (b) if r, s in $\mathbb{Q}$, then $b^{r\cdot s} = (b^r)^s$.

2. For $x \in \mathbb{R}$, let $B(x) = \{b^t \in \mathbb{R} \,|\, t \in \mathbb{Q}, t \leqslant x\}$. Show that $\sup B(x)$ exists for all $x \in \mathbb{R}$, and $b^r = \sup B(r)$ if $r \in \mathbb{Q}$.

3. Define $b^x = \sup B(x)$ for $x \in \mathbb{R}$. Show that $B(x) > 0$ for all $x \in \mathbb{R}$ and the law of exponents (for exponents in $\mathbb{R}$)

   (a) if x, y in $\mathbb{R}$, then $b^{x+y} = b^x \cdot b^y$, (b) if $x, y > 0$, then $b^{x\cdot y} = (b^x)^y$,

   are also valid.

4. Show that if $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$, then $b^{x_1} < b^{x_2}$. This implies that if x_1, x_2 are two numbers in $\mathbb{R}$ satisfying $b^{x_1} = b^{x_2}$, then $x_1 = x_2$.

5. Let $y > 0$ be given. Show that if $u, v \in \mathbb{R}$ such that $b^u < y$ and $b^v > y$, then $b^{u+1/n} < y$ and $b^{v-1/n} > y$ for sufficiently large n.

6. Let $y > 0$ be given, and $A \subseteq \mathbb{R}$ be the set of all w such that $b^w < y$. Show that $\sup A$ exists and $x = \sup A$ satisfies $b^x = y$. The number x (the uniqueness is guaranteed by 4) satisfying $b^x = y$ is called the logarithm of y to the base b, and is denoted by $\log_b y$.

Hint: Make use of Problem 3 in Exercise 2.

Proof. We note that $\mathbb{R}$ satisfies Archimedean property and the least upper bound property.

1. Note that the exponential law holds if the exponents are integers; that is,

$$b^{n+m} = b^n \cdot b^m \quad \text{and} \quad b^{nm} = (b^n)^m \qquad \forall\, n, m \in \mathbb{Z}\,.$$

For $m, n \in \mathbb{N}$, we "define" $b^{\frac{n}{m}}$ as the n-th power of $b^{\frac{1}{m}}$; that is, $b^{\frac{n}{m}} = \big(b^{\frac{1}{m}}\big)^n$. Then for $m, n \in \mathbb{N}$,

$$\big[\big(b^{\frac{1}{m}}\big)^n\big]^m = \big(b^{\frac{1}{m}}\big)^{mn} = b^{\frac{mn}{m}} = b^n$$

which implies that $\big(b^{\frac{1}{m}}\big)^n$ is the m-th root of b^n if $m, n \in \mathbb{N}$. Moreover, $\big(b^{\frac{1}{mn}}\big)^n = b^{\frac{1}{m}}$ and $\big(b^{\frac{1}{mn}}\big)^m = b^{\frac{1}{n}}$; thus we establish that

$$b^{\frac{n}{m}} = \big(b^{\frac{1}{m}}\big)^n = (b^n)^{\frac{1}{m}} \qquad \text{and} \qquad b^{\frac{1}{mn}} = \big(b^{\frac{1}{m}}\big)^{\frac{1}{n}} \qquad \forall\, m, n \in \mathbb{N}\,. \tag{♠}$$

Suppose that $r = \dfrac{q_1}{p_1}$ and $s = \dfrac{q_2}{p_2}$, where $p_1, p_2, q_1, q_2 \in \mathbb{N}$. Then (♠) implies that

$$(b^r)^s = \big(b^{\frac{q_1}{p_1}}\big)^{\frac{q_2}{p_2}} = \big(b^{\frac{1}{p_1}}\big)^{\frac{q_1 q_2}{p_2}} = \Big[\big(b^{\frac{1}{p_1}}\big)^{\frac{1}{p_2}}\Big]^{q_1 q_2} = \big(b^{\frac{1}{p_1 p_2}}\big)^{q_1 q_2} = b^{\frac{q_1 q_2}{p_1 p_2}}$$

and

$$b^{r+s} = b^{\frac{p_2 q_1 + p_1 q_2}{p_1 p_2}} = \big(b^{\frac{1}{p_1 p_2}}\big)^{p_2 q_1 + p_1 q_2} = \big(b^{\frac{1}{p_1 p_2}}\big)^{p_2 q_1} \cdot \big(b^{\frac{1}{p_1 p_2}}\big)^{p_1 q_2} = b^{\frac{p_2 q_1}{p_1 p_2}} \cdot b^{\frac{p_1 q_2}{p_1 p_2}} = b^r \cdot b^s\,.$$

Therefore,

$$b^{r+s} = b^r \cdot b^s \quad \text{and} \quad b^{rs} = (b^r)^s \qquad \forall\, r, s \in \mathbb{Q} \;\text{ and }\; r, s > 0\,. \tag{♡}$$

For $r \in \mathbb{Q}$ and $r < 0$, we define $b^r = (b^{-r})^{-1}$. Then if $r, s \in \mathbb{Q}$ and $r, s < 0$, we have

$$b^{r+s} = (b^{-(r+s)})^{-1} = (b^{-r} \cdot b^{-s})^{-1} = (b^{-r})^{-1} \cdot (b^{-s})^{-1} = b^r \cdot b^s$$

and

$$(b^r)^s = \big[(b^{-r})^{-1}\big]^s\,.$$

2. First we show that $x \in \mathbb{R}$, $B(x)$ is non-empty and bounded from above. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that $-x < n$. Therefore, there exists a rational number $-n$ such that $-n < x$; thus $b^{-n} \in B(x)$ which implies that $B(x)$ is non-empty.

   On the other hand, the Archimedean Property implies that there exists $m \in \mathbb{N}$ such that $x < m$. By the fact that

   $$b^t \leqslant b^s \qquad \text{whenever} \quad t \leqslant s \text{ and } t, s \in \mathbb{Q}\,, \tag{$*$}$$

   we conclude that b^m is an upper bound for $B(x)$. Therefore, $B(x)$ is bounded from above. By the least upper bound property, we conclude that $\sup B(x)$ exists for all $x \in \mathbb{R}$.

   Next we show that $b^r = \sup B(r)$ if $r \in \mathbb{Q}$. To see this, we note that $b^r \in B(r)$ if $r \in \mathbb{Q}$. On the other hand, $(*)$ implies that b^r is an upper bound for $B(r)$; thus $\sup B(r) = b^r$.

3. We first show that

   $$\sup(cA) = c \cdot \sup A \qquad \forall\, c > 0\,, \tag{$\star$}$$

   where $cA = \big\{c \cdot x \,\big|\, x \in A\big\}$. To see $(\star)$, we observe that

   $$x \in A \Rightarrow x \leqslant \sup A \Rightarrow c \cdot x \leqslant c \cdot \sup A \;\; \text{(by the compatibility of $\cdot$ and $\leqslant$)}\,;$$

   thus every element in cA is bounded from above by $c \cdot \sup A$. Therefore,

   $$\sup(cA) \leqslant c \cdot \sup A\,.$$

   On the other hand, let $\varepsilon > 0$ be given. Then there exists $x \in A$ and $x > \sup A - \dfrac{\varepsilon}{c}$. Therefore, $c \cdot x > c \cdot \sup A - \varepsilon$; thus

   $$\sup(cA) \geqslant c \cdot x > c \cdot \sup A - \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we find that $\sup(cA) \geqslant c \cdot \sup A$; thus $(\star)$ is concluded.

Next we show that

$$\sup \{b^t \,|\, t \in \mathbb{Q}, t \leqslant x\} = \inf \{b^s \,|\, s \in \mathbb{Q}, s \geqslant x\} . \tag{$\diamond$}$$

Let $S(x) = \{b^s \,|\, s \in \mathbb{Q}, s \geqslant x\}$. If $b^t \in B(x)$, then b^t is a lower bound for $S(x)$. Therefore, $B(x)$ is a subset of the collection of all lower bounds for $S(x)$. By Problem 3 of Exercise 2,

$$\sup B(x) \leqslant \sup \{y \,|\, y \text{ is a lower bound for } S(x)\} = \inf S(x) .$$

Suppose that $\sup B(x) < \inf S(x)$. Since $b^{\frac{1}{n}} \searrow 1$ as $n \to \infty$ (Problem 3 of Exercise 2), there exists $n \in \mathbb{N}$ such that $\inf S(x) > b^{\frac{1}{n}} \sup B(x)$. By the fact that there exists $r \in \mathbb{Q}$ and $x \leqslant r \leqslant x + \dfrac{1}{n}$, we find that

$$\begin{aligned}\inf S(x) > b^{\frac{1}{n}} \sup B(x) &= \sup \{b^{r+\frac{1}{n}} \,|\, r \in \mathbb{Q}, r \leqslant x\} = \sup \Big\{b^s \,\Big|\, s \in \mathbb{Q}, s \leqslant x + \frac{1}{n}\Big\} \\ &\geqslant b^r \geqslant \inf \{b^s \,|\, s \in \mathbb{Q}, s \geqslant x\} = \inf S(x) ,\end{aligned}$$

a contradiction. Observe that

$$\sup A^{-1} = \big(\inf A\big)^{-1} \qquad \text{for every subset } A \text{ of } (0,\infty) ,$$

where $A^{-1} = \{t^{-1} \,|\, t \in A\}$ and $(0,\infty)$ is the collection consisting of positive elements in $\mathbb{R}$. Therefore, $(\diamond)$ implies that for $x \in \mathbb{R}$,

$$\begin{aligned}b^{-x} &= \sup \{b^t \,|\, t \in \mathbb{Q}, t \leqslant -x\} = \sup \{b^{-t} \,|\, t \in \mathbb{Q}, t \geqslant x\} = \Big[\inf \{b^t \,|\, t \in \mathbb{Q}, t \geqslant x\}\Big]^{-1} \\ &= (b^x)^{-1} .\end{aligned}$$

Now we show the law of exponential

$$b^x \cdot b^y = b^{x+y} \qquad \forall\, x, y \in \mathbb{R} . \tag{$\star\star$}$$

Let $x, y \in \mathbb{R}$ be given. If $t, s \in \mathbb{Q}$ and $t \leqslant x$, $s \leqslant y$, then $t + s \in \mathbb{Q}$ and $t + s \leqslant x + y$; thus

$$b^t \cdot b^s = b^{t+s} \leqslant \sup B(x+y) = b^{x+y} .$$

For any given rational $t \leqslant x$, taking the supremum of the left-hand side over all rational $s \leqslant y$ and using $(\star)$ we find that

$$\begin{aligned}b^{-x} &= \sup \{b^t \,|\, t \in \mathbb{Q}, t \leqslant -x\} = \sup \{b^{-t} \,|\, t \in \mathbb{Q}, t \geqslant x\} = \Big[\inf \{b^t \,|\, t \in \mathbb{Q}, t \geqslant x\}\Big]^{-1} \\ &= (b^x)^{-1} .\end{aligned}$$

Taking the supremum of the left-hand side over all rational $t \leqslant x$, using $(\star)$ again we find that

$$b^y \cdot b^x = b^y \cdot \sup \{b^t \,|\, t \in \mathbb{Q}, t \leqslant x\} = \sup \{b^{t+y} \,|\, t \in \mathbb{Q}, t \leqslant x\} \leqslant b^{x+y} ;$$

thus we establish that

$$b^x \cdot b^y \leqslant b^{x+y} \qquad \forall\, x, y \in \mathbb{R}\,. \tag{$\diamond\diamond$}$$

Now, note that ($\diamond\diamond$) implies that for all $x, y \in \mathbb{R}$,

$$b^y = b^{-x+x+y} \geqslant b^{-x} \cdot b^{x+y} = (b^x)^{-1} \cdot b^{x+y} \geqslant (b^x)^{-1} \cdot b^x \cdot b^y = b^y\,.$$

The inequality above is indeed an equality and we obtain that

$$b^y = b^{-x} b^{x+y} \qquad \forall\, x, y \in \mathbb{R}\,.$$

This is indeed ($\star\star$) because of that $b^{-x} = (b^x)^{-1}$.

Next we show that $(b^x)^y = \sup B(x \cdot y)$ for all $x > 0$ and $y \in \mathbb{R}$. For $z > 0$, define $A(z) = \big\{s \in \mathbb{R} \,\big|\, s \in \mathbb{Q}, 0 < s \leqslant z\big\}$. Note that if $z > 0$, then $b^z = \sup A(z)$. Since for $x > 0$, we have $b^x > 1$; thus for $x, y > 0$,

$$(b^x)^y = \sup\big\{(b^x)^t \,\big|\, t \in \mathbb{Q}, 0 < t \leqslant y\big\} = \sup_{t \in A(y)} (b^x)^t = \sup_{t \in A(y)} \Big(\sup_{s \in A(x)} b^s\Big)^t\,.$$

By Problem 5 of Exercise 2,

$$\sup_{t \in A(y)} \Big(\sup_{s \in A(x)} b^s\Big)^t = \sup_{(t,s) \in A(y) \times A(x)} (b^s)^t = \sup_{(t,s) \in A(y) \times A(x)} b^{st} = b^{\sup_{(t,s) \in A(y) \times A(x)} ts} = b^{xy}\,.$$

4. Let $x_1 < x_2$ be given. Then **AP** implies that there exists $r, s \in \mathbb{Q}$ such that $x_1 < r < s < x_2$. Therefore, $B(x_1) \subseteq B(r) \subseteq B(s) \subseteq B(x_2)$; thus

$$b^{x_1} = \sup B(x_1) \leqslant \sup B(r) \leqslant \sup B(s) \leqslant \sup B(x_2) = b^{x_2}\,.$$

Since $B(r) = b^r$ and $B(s) = b^s$, we must have $B(r) < B(s)$; thus 4 is concluded.

5. Since $\dfrac{y}{b^u} > 1$ and $\dfrac{b^v}{y} > 1$, by the fact that $b^{\frac{1}{n}} \to 1$ as $n \to \infty$, there exist $N_1, N_2 > 0$ such that

$$\big|b^{\frac{1}{n}} - 1\big| < \frac{y}{b^u} - 1 \;\text{ whenever }\; n \geqslant N_1 \quad\text{and}\quad \big|b^{\frac{1}{n}} - 1\big| < \frac{b^v}{y} - 1 \;\text{ whenever }\; n \geqslant N_2\,.$$

Let $N = \max\{N_1, N_2\}$. For $n \geqslant N$, we have $b^{\frac{1}{n}} < \dfrac{y}{b^u}$ and $b^{\frac{1}{n}} < \dfrac{b^v}{y}$ or equivalently,

$$b^{u+\frac{1}{n}} < y \quad\text{and}\quad b^{v-\frac{1}{n}} > y \qquad \forall\, n \geqslant N\,.$$

6. Let $A = \big\{w \in \mathbb{R} \,\big|\, b^w < y\big\}$. Since $b > 1$, 2 of Problem 3 in Exercise 2 implies that

$$b^n > 1 + n(b-1) \quad\text{ whenever }\quad n \geqslant 2\,. \tag{$\star\star\star$}$$

By **AP**, there exists $N \geqslant 2$ such that $1 + N(b-1) > y$; thus A is bounded from above by N. Moreover, there exists $M \geqslant 2$ such that

$$1 + M(b-1) > \frac{1}{y}\,;$$

thus ($\star\star\star$) implies that $b^{-M} < y$ or $-M \in A$. Therefore, A is non-empty. By **LUBP**, we conclude that $\sup A$ exists.

Let $x = \sup A$. Then $x + \dfrac{1}{n} \notin A$; thus $b^{x+\frac{1}{n}} \geqslant y$ for all $n \in \mathbb{N}$. Since $b^{\frac{1}{n}} \to 1$ sa $n \to \infty$, we find that

$$b^x = b^x \lim_{n\to\infty} b^{\frac{1}{n}} = \lim_{n\to\infty} b^{x+\frac{1}{n}} \geqslant y\,.$$

On the other hand, 4 implies that $x - \dfrac{1}{n} \in A$; thus $b^{x-\frac{1}{n}} < y$ for all $n \in \infty$ and we have

$$b^x = b^x \lim_{n\to\infty} b^{-\frac{1}{n}} = \lim_{n\to\infty} b^{x-\frac{1}{n}} \leqslant y\,.$$

Therefore, $b^x = y$. □

Problem 2. In this problem we prove the Intermediate Value Theorem:

> Let $f : [a, b] \to \mathbb{R}$ be continuous (at every point of $[a, b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) \qquad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq [a, b].$$
>
> If $f(a)f(b) < 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

Complete the following.

1. W.L.O.G, we can assume that $f(a) < 0$. Define the set $S = \{x \in [a, b] \,|\, f(x) > 0\}$. Show that $\inf S$ exists.

2. Let $c = \inf S$. Show that $f(c) \geqslant 0$.

3. Conclude that $f(c) \leqslant 0$ as well.

Hint:

1. Show that S is non-empty and bounded from below.

2. Show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in S such that $c_n \to c$ as $n \to \infty$.

3. Show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in $[a, c)$ such that $c_n \to c$ as $n \to \infty$.

Proof. 1. Since $f(b) > 0$, $b \in S$. Moreover, a is a lower bound for S; thus S is non-empty and bounded from below. By the completeness of $\mathbb{R}$, $\inf S \in \mathbb{R}$ exists.

2. Let $c = \inf S$. For each $n \in \mathbb{N}$, there exists $c_n < c + \dfrac{1}{n}$ and $c_n \in S$. Then $f(c_n) > 0$ for all $n \in \mathbb{N}$ and

$$c \leqslant c_n < c + \frac{1}{n} \qquad \forall\, n \in \mathbb{N}\,.$$

Then the Sandwich Lemma implies that $c_n \to c$ as $n \to \infty$. By the continuity of f,

$$f(c) = f\big(\lim_{n\to\infty} c_n\big) = \lim_{n\to\infty} f(c_n) \geqslant 0\,.$$

3. By 2, $a \neq c$. Consider the sequence $\{c_n\}_{n=1}^{\infty}$ defined by $c_n = c - \dfrac{c-a}{n}$. Then $\{c_n\}_{n=1}^{\infty} \subseteq [a, c)$. Moreover, by the fact that $c = \inf S$ and $c_n < c$, $c_n \notin S$ for all $n \in \mathbb{N}$. Therefore, $f(c_n) \leqslant 0$ for all $n \in \mathbb{N}$. Since $c_n \to c$ as $n \to \infty$, by the continuity of f we find that

$$f(c) = f\big(\lim_{n\to\infty} c_n\big) = \lim_{n\to\infty} f(c_n) \leqslant 0\,. \qquad \square$$

Problem 3. In this problem we prove the Extreme Value Theorem:

> Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be continuous (at every point of $[a, b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big) \qquad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty} \subseteq [a, b].$$
>
> Then there exist $c, d \in [a, b]$ such that $f(c) = \sup\limits_{x\in[a,b]} f(x)$ and $f(d) = \inf\limits_{x\in[a,b]} f(x)$.

Complete the following.

1. Show that there exist sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ in $[a, b]$ such that

$$\lim_{n\to\infty} f(c_n) = \sup_{x\in[a,b]} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} f(d_n) = \inf_{x\in[a,b]} f(x)\,.$$

2. Extract convergent subsequences $\{c_{n_k}\}_{k=1}^{\infty}$ and $\{d_{n_k}\}_{k=1}^{\infty}$ with limit c and d, respectively. Show that $c, d \in [a, b]$.

3. Show that $f(c) = \sup\limits_{x\in[a,b]} f(x)$ and $f(d) = \inf\limits_{x\in[a,b]} f(x)$.

Proof. It suffices to show the case of $\sup\limits_{x\in[a,b]} f(x)$ since $\inf\limits_{x\in[a,b]} f(x) = -\sup\limits_{x\in[a,b]} (-f)(x)$ by Problem 2 of Exercise 3.

1. We first show that $f([a, b])$ is bounded. Suppose the contrary that $f([a, b])$ is not bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Since $\{x_n\}_{n=1}^{\infty} \subseteq [a, b]$, $\{x_n\}_{n=1}^{\infty}$ is bounded. By the fact that **MSP** $\Rightarrow$ **BWP**, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. By the continuity of f, $\{f(x_{n_k})\}_{k=1}^{\infty}$ is also convergent; thus Proposition 1.39 in the lecture note implies that $\{f(x_{n_k})\}_{k=1}^{\infty}$ is bounded, a contradiction to that $|f(x_{n_k})| \geqslant n_k \geqslant k$ for all $k \in \mathbb{N}$.

   Since $f([a, b])$ is bounded, $M = \sup f([a, b]) = \sup\limits_{x\in[a,b]} f(x)$ exists. For each $n \in \mathbb{R}$, there exists $c_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(c_n) \leqslant M\,.$$

   By the Sandwich Lemma, $\lim\limits_{n\to\infty} f(c_n) = M = \sup\limits_{x\in[a,b]} f(x)$.

2. Since $\{c_n\}_{n=1}^{\infty} \subseteq [a, b]$, $\{c_n\}_{n=1}^{\infty}$ is bounded. By the fact that **MSP** $\Rightarrow$ **BWP**, there exists a convergent subsequence $\{c_{n_k}\}_{k=1}^{\infty}$ of $\{c_n\}_{n=1}^{\infty}$ with limit c. Since $a \leqslant c_{n_k} \leqslant b$ for all $k \in \mathbb{N}$, by a Proposition that we talked about in class we conclude that $a \leqslant c \leqslant b$.

3. Since $c_{n_k} \to c$ as $k \to \infty$, the continuity of f implies that

$$f(c) = f(\lim_{k\to\infty} c_{n_k}) = \lim_{k\to\infty} f(c_{n_k}) = \sup_{x\in[a,b]} f(x)\,. \qquad \square$$

Problem 4. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in $\mathbb{R}$. Prove the following inequalities:

$$\begin{aligned}
\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leqslant \liminf_{n\to\infty}(x_n+y_n) \leqslant \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
&\leqslant \limsup_{n\to\infty}(x_n+y_n) \leqslant \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\,; \\
\big(\liminf_{n\to\infty}|x_n|\big)\big(\liminf_{n\to\infty}|y_n|\big) &\leqslant \liminf_{n\to\infty}|x_ny_n| \leqslant \big(\liminf_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big) \\
&\leqslant \limsup_{n\to\infty}|x_ny_n| \leqslant \big(\limsup_{n\to\infty}|x_n|\big)\big(\limsup_{n\to\infty}|y_n|\big)\,.
\end{aligned}$$

Give examples showing that the equalities are generally not true.

Proof. 1. Let $k \in \mathbb{N}$ be fixed. Note that for $n \geqslant k$, we have

$$\inf_{n\geqslant k}(x_n+y_n) \leqslant x_n + y_n \leqslant \sup_{n\geqslant k}(x_n+y_n)\,.$$

Note that the LHS and the RHS are functions of k and is independent of n. Therefore,

$$\inf_{n\geqslant k}\Big[\inf_{n\geqslant k}(x_n+y_n) - y_n\Big] \leqslant \inf_{n\geqslant k} x_n \leqslant \inf_{n\geqslant k}\Big[\sup_{n\geqslant k}(x_n+y_n) - y_n\Big]$$

which further shows that

$$\inf_{n\geqslant k}(x_n+y_n) - \sup_{n\geqslant k} y_n \leqslant \inf_{n\geqslant k} x_n \leqslant \sup_{n\geqslant k}(x_n+y_n) - \sup_{n\geqslant k} y_n\,.$$

Therefore,

$$\inf_{n\geqslant k}(x_n+y_n) \leqslant \inf_{n\geqslant k} x_n + \sup_{n\geqslant k} y_n \leqslant \sup_{n\geqslant k}(x_n+y_n) \qquad \forall\, k \in \mathbb{N}\,,$$

and the first inequality follows from the fact that

$$\inf_{n\geqslant k} x_n + \inf_{n\geqslant k} y_n \leqslant \inf_{n\geqslant k}(x_n+y_n) \leqslant \inf_{n\geqslant k} x_n + \sup_{n\geqslant k} y_n \leqslant \sup_{n\geqslant k}(x_n+y_n) \leqslant \sup_{n\geqslant k} x_n + \sup_{n\geqslant k} y_n$$

for each $k \in \mathbb{N}$.

2. Let $k \in \mathbb{N}$ be fixed. Note that for $n \geqslant k$, we have

$$\inf_{n\geqslant k}\Big[|x_n|\big(|y_n| + \frac{1}{k}\big)\Big] \leqslant |x_n|\big(|y_n|+\frac{1}{k}\big) \leqslant \sup_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]\,.$$

Note that the LHS and the RHS for functions of k and is independent of n. Therefore,

$$\inf_{n\geqslant k}\frac{\inf_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{|y_n|+\frac{1}{k}} \leqslant \inf_{n\geqslant k}|x_n| \leqslant \inf_{n\geqslant k}\frac{\sup_{n\geqslant k}\Big[|x_n|\big(|y_n|+\frac{1}{k}\big)\Big]}{|y_n|+\frac{1}{k}}\,.$$

By the fact that

$$\inf_{n\geqslant k} \frac{1}{|y_n| + \frac{1}{k}} = \frac{1}{\sup\limits_{n\geqslant k} \big(|y_n| + \frac{1}{k}\big)},$$

we find that

$$\frac{\inf\limits_{n\geqslant k} \big[|x_n|\big(|y_n| + \frac{1}{k}\big)\big]}{\sup\limits_{n\geqslant k} \big(|y_n| + \frac{1}{k}\big)} \leqslant \inf_{n\geqslant k} |x_n| \leqslant \inf_{n\geqslant k} \frac{\sup\limits_{n\geqslant k} \big[|x_n|\big(|y_n| + \frac{1}{k}\big)\big]}{\sup\limits_{n\geqslant k} \big(|y_n| + \frac{1}{k}\big)};$$

thus

$$\inf_{n\geqslant k} \big[|x_n|\big(|y_n| + \frac{1}{k}\big)\big] \leqslant \inf_{n\geqslant k} |x_n| \sup_{n\geqslant k} \big(|y_n| + \frac{1}{k}\big) \leqslant \sup_{n\geqslant k} \big[|x_n|\big(|y_n| + \frac{1}{k}\big)\big].$$

The second inequality follows from the fact that

$$\begin{aligned}\inf_{n\geqslant k} |x_n| \inf_{n\geqslant k} \big(|y_n| + \frac{1}{k}\big) \leqslant \inf_{n\geqslant k} \big[|x_n|\big(|y_n| + \frac{1}{k}\big)\big] &\leqslant \inf_{n\geqslant k} |x_n| \sup_{n\geqslant k} \big(|y_n| + \frac{1}{k}\big) \\ \leqslant \sup_{n\geqslant k} \big[|x_n|\big(|y_n| + \frac{1}{k}\big)\big] &\leqslant \sup_{n\geqslant k} |x_n| \sup_{n\geqslant k} \big(|y_n| + \frac{1}{k}\big)\end{aligned}$$

for each $k \in \mathbb{N}$, and passing to the limit as $k \to \infty$.

3. Let $x_n = 2 + \sin n$ and $y_n = 2 + \cos n$. Then $x_n, y_n > 0$, and

$$\liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n = 1, \quad \limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n = 3.$$

By Problem 3, the set $\big\{x \in [0, 2\pi] \,\big|\, x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\big\}$ is dense in $[0, 2\pi]$; thus for each $\theta \in [0, 2\pi]$ there exists an increasing sequence $\{k_j\}_{j=1}^\infty \subseteq \mathbb{N}$ such that $x_{k_j} = k_j \pmod{2\pi}$ and $\big\{x_{k_j}\big\}_{j=1}^\infty$ converges to θ. This implies that for each $\theta \in [-1, 1]$, there exists a subsequence $\{\cos k_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \cos n_j = \cos\theta \quad \text{and} \quad \lim_{j\to\infty} \sin n_j = \sin\theta.$$

Therefore, we have

$$\liminf_{n\to\infty} (x_n + y_n) = 4 - \sqrt{2}, \quad \limsup_{n\to\infty} (x_n + y_n) = 4 + \sqrt{2},$$

and

$$\liminf_{n\to\infty} x_n y_n = \frac{9}{2} - 2\sqrt{2}, \quad \limsup_{n\to\infty} x_n y_n = \frac{9}{2} + 2\sqrt{2}.$$

Therefore,

$$\begin{aligned}\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n < \liminf_{n\to\infty} (x_n + y_n) &< \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\ < \limsup_{n\to\infty} (x_n + y_n) &< \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty} x_n \cdot \liminf_{n\to\infty} y_n < \liminf_{n\to\infty} (x_n y_n) &< \liminf_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n \\ < \limsup_{n\to\infty} (x_n y_n) &< \limsup_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n.\end{aligned}$$

Therefore, the equalities are generally not true. □

Problem 5. Prove that

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \leqslant \liminf_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}.$$

Give examples to show that the equalities are not true in general. Is it true that $\lim\limits_{n\to\infty} \sqrt[n]{|x_n|}$ exists implies that $\lim\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$ also exists?

Proof. W.L.O.G. we can assume that $\liminf\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|} > 0$ and $\limsup\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|} < \infty$. Let $a = \liminf\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$ and $b = \limsup\limits_{n\to\infty} \dfrac{|x_{n+1}|}{|x_n|}$, and $\varepsilon > 0$ be given such that $a - \varepsilon > 0$. Then there exists $N > 0$ such that

$$a - \varepsilon < \frac{|x_{n+1}|}{|x_n|} < b + \varepsilon \qquad \forall\, n \geqslant N.$$

Therefore,

$$(a-\varepsilon)|x_n| < |x_{n+1}| < (b+\varepsilon)|x_n| \qquad \forall\, n \geqslant N$$

which implies that if $n > N$,

$$|x_n| > (a-\varepsilon)|x_{n-1}| > (a-\varepsilon)^2|x_{n-2}| > \cdots > (a-\varepsilon)^{n-N}|x_N|$$

and

$$|x_n| < (b+\varepsilon)|x_{n-1}| < (b+\varepsilon)^2|x_{n-2}| < \cdots < (b+\varepsilon)^{n-N}|x_N|.$$

The inequality above implies that

$$(a-\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|} < \sqrt[n]{|x_n|} < (b+\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|};$$

thus

$$\liminf_{n\to\infty} \Big[(a-\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|}\Big] \leqslant \liminf_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \Big[(b+\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|}\Big].$$

By Problem 3 of Exercise 2, $\lim\limits_{n\to\infty} b^{\frac{1}{n}} = 1$ for all $b > 0$. Therefore,

$$\liminf_{n\to\infty} \Big[(a-\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|}\Big] = \lim_{n\to\infty} (a-\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|} = a - \varepsilon = \liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} - \varepsilon$$

and

$$\limsup_{n\to\infty} \Big[(b+\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|}\Big] = \lim_{n\to\infty} (b+\varepsilon)^{1-\frac{N}{n}} \sqrt[n]{|x_N|} = b + \varepsilon = \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} + \varepsilon.$$

Since the inequality above holds for all $\varepsilon > 0$, we conclude that

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \leqslant \liminf_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \sqrt[n]{|x_n|} \leqslant \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a real sequence defined by

$$x_n = \begin{cases} 2^{-n} & \text{if } n \text{ is odd}, \\ 4^{-n} & \text{if } n \text{ is even}, \end{cases}$$

or $x_n = (3+(-1)^n)^{-n}$. Then $\sqrt[n]{|x_n|} = 3+(-1)^n$ which shows that

$$\liminf_{n\to\infty} \sqrt[n]{|x_n|} = \frac{1}{4} \qquad \text{and} \qquad \limsup_{n\to\infty} \sqrt[n]{|x_n|} = \frac{1}{2}\,.$$

To compute the limit superior and limit inferior of $\dfrac{|x_{n+1}|}{|x_n|}$, we define

$$y_n = \frac{|x_{n+1}|}{|x_n|} = \frac{(3+(-1)^{n+1})^{-n-1}}{(3+(-1)^n)^{-n}} = \frac{1}{3-(-1)^n}\Big(\frac{3-(-1)^n}{3+(-1)^n}\Big)^{-n}$$

and observe that $\lim\limits_{n\to\infty} y_{2n} = 0$ and $\lim\limits_{n\to\infty} y_{2n+1} = \infty$. Since $y_n \in [0,\infty)$, we conclude that 0 is the smallest cluster point of $\{y_n\}_{n=1}^{\infty}$ and ∞ is the largest "cluster point" of $\{y_n\}_{n=1}^{\infty}$. This shows that

$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = 0 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \infty\,. \qquad \square$$

Problem 6. Given the following sets consisting of elements of some sequence of real numbers. Find the limsup and liminf of the sequence.

1. $\big\{\cos m \,\big|\, m = 0,1,2,\cdots\big\}$.
2. $\big\{\sqrt[m]{|\sin m|} \,\big|\, m = 1,2,\cdots\big\}$.
3. $\Big\{\Big(1+\dfrac{1}{m}\Big)\sin\dfrac{m\pi}{6} \,\Big|\, m = 1,2,\cdots\Big\}$.

Hint: 1. First show that for all irrational α, the set

$$S = \big\{x \in [0,1] \,\big|\, x = k\alpha \pmod 1 \text{ for some } k \in \mathbb{N}\big\}$$

is dense in $[0,1]$; that is, for all $y \in [0,1]$ and $\varepsilon > 0$, there exists $x \in S \cap (y-\varepsilon, y+\varepsilon)$. Then choose $\alpha = \dfrac{1}{2\pi}$ to conclude that

$$T = \big\{x \in [0,2\pi] \,\big|\, x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\big\}$$

is dense in $[0,2\pi]$. To prove that S is dense in $[0,1]$, you might want to consider the following set

$$S_k = \big\{x \in [0,1] \,\big|\, x = \ell\alpha \pmod 1 \text{ for some } 1 \leqslant \ell \leqslant k+1\big\}$$

Note that there must be two points in S_k whose distance is less than $\dfrac{1}{k}$. What happened to (the multiples of) the difference of these two points?
2. Use the fact that π is a Liouville number; that is, there exists $d \in \mathbb{N}$ such that

$$\Big|\pi - \frac{p}{q}\Big| \geqslant \frac{1}{q^d} \qquad \forall\, p,q \in \mathbb{Z}, q \neq 0\,.$$

Proof. 1. Define $S_k = \big\{x \in [0,1] \,\big|\, x = \ell\alpha \pmod 1 \text{ for some } 1 \leqslant \ell \leqslant k+1\big\}$. Let $1 \leqslant \ell_1, \ell_2 \leqslant k+1$, and $x, y \in [0,1]$ satisfying that $x = \ell_1\alpha \pmod 1$ and $y = \ell_2\alpha \pmod 1$. Then by the fact that $\alpha \notin \mathbb{Q}$,

$$x = y \quad \Leftrightarrow \quad \ell_1\alpha = \ell_2\alpha \pmod 1 \quad \Leftrightarrow \quad (\ell_1-\ell_2)\alpha \in \mathbb{Z} \quad \Leftrightarrow \quad \ell_1 - \ell_2 = 0\,.$$

Therefore, there are $(k+1)$ distinct points in S_k (this also shows that each $k \in \mathbb{N}$ corresponds to different point $x = k\alpha \pmod 1$ in S). Moreover, $x \notin \mathbb{Q}$ if $x \in S_k$. By the pigeonhole principle, there exist x, y in S_k satisfying that $0 < |x - y| < \frac{1}{k}$.

Let $\varepsilon > 0$ be given. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. By the discussion above, there exist $x, y \in S_n$ such that $0 < |x - y| < \varepsilon$. Suppose that $x = n_1\alpha \pmod 1$ and $y = n_2\alpha$ (mod 1), and define $m = |n_1 - n_2|$. The point $z \in [0,1]$ satisfying $z = m\alpha \pmod 1$ has the property that $z \in (0, \varepsilon) \cup (1 - \varepsilon, 1)$. Therefore,

$$(\forall\, \varepsilon > 0)(\exists\, x \in S)\big(x \in (0,\varepsilon) \cup (1-\varepsilon, 1)\big)\,.$$

Let $y \in [0,1]$ and $\varepsilon > 0$ be given. The discussion above provides an $x \in (0,1)$ such that $x = k\alpha$ (mod 1) for some $k \in \mathbb{N}$ and $x \in (0,\varepsilon) \cup (1-\varepsilon, 1)$. Then some constant multiple of x must belong to $(y - \varepsilon, y + \varepsilon)$. If $\ell x \in (y - \varepsilon, y + \varepsilon)$, then $z = k\ell\alpha \pmod 1$ in $(y - \varepsilon, y + \varepsilon)$. This shows that S is dense in $[0,1]$.

Having established that S is dense in $[0,1]$, we find that T is dense in $[0, 2\pi]$. Therefore, for each $\theta \in [0, 2\pi]$ there exists an increasing sequence $\{m_j\}_{j=1}^\infty \subseteq \mathbb{N}$ such that $x_{m_j} = m_j$ (mod 2π) and $\{x_{m_j}\}_{j=1}^\infty \subseteq [0, 2\pi]$ converges to θ. In particular, for each $\theta \in [0, 2\pi]$ there exists an increasing sequence $\{m_j\}_{j=1}^\infty \subseteq \mathbb{N}$ such that

$$\lim_{j\to\infty} \cos m_j = \cos\theta \qquad \text{and} \qquad \lim_{j\to\infty} \sin m_j = \sin\theta\,;$$

thus we conclude that $\limsup\limits_{m\to\infty} \cos m = 1$ and $\liminf\limits_{m\to\infty} \cos m = -1$.

2. Since π is not a Liouville number, there exists $d \in \mathbb{N}$ such that

$$\Big|\pi - \frac{p}{q}\Big| \geqslant \frac{1}{q^d} \qquad \forall\, p, q \in \mathbb{Z}, q \neq 0\,. \tag{0.1}$$

For each $m \in \mathbb{N}$, let $q_m \in \mathbb{N}$ be such that

$$\inf_{q\in\mathbb{N}} |q\pi - m| = |q_m\pi - m|\,. \tag{0.2}$$

Such q_m exists since the infimum indeed occurs in a finite set of $\mathbb{N}$. Using (0.1), we find that

$$\frac{1}{q_m^{d-1}} \leqslant |q_m\pi - m| \qquad \forall\, m \in \mathbb{N}\,.$$

On the other hand, because of (0.2) we must have

$$|q_m\pi - m| \leqslant \frac{\pi}{2} \qquad \forall\, m \gg 1 \ \ \text{(in fact, } m \geqslant 6 \text{ is enough)}$$

since we cannot have $|q_m\pi - m| > \frac{\pi}{2}$, $|(q_m+1)\pi - m| > \frac{\pi}{2}$ and $|(q_m-1)\pi - m| > \frac{\pi}{2}$ simultaneously. Therefore,

$$\frac{1}{q_m^{d-1}} \leqslant |q_m\pi - m| \leqslant \frac{\pi}{2} \qquad \forall\, m \gg 1 \tag{0.3}$$

which, together with the inequality $\frac{2}{\pi}x \leqslant \sin x$ for all $x \in \left[0, \frac{\pi}{2}\right]$, further shows that

$$\frac{2}{\pi}\frac{1}{q_m^{d-1}} \leqslant \sin\frac{1}{q_m^{d-1}} \leqslant |\sin m| \leqslant 1 \qquad \forall\, m \gg 1\,. \tag{0.4}$$

The inequality above shows that

$$\Big(\frac{2}{\pi q_m^{d-1}}\Big)^{\frac{1}{m}} \leqslant \sqrt[m]{|\sin m|} \leqslant 1 \qquad \forall\, m \gg 1\,.$$

Since (0.3) implies that $\frac{m}{\pi} - \frac{1}{2} \leqslant q_m \leqslant \frac{m}{\pi} + \frac{1}{2}$ for all $m \gg 1$, the fact that

$$\lim_{m\to\infty}\Big(\frac{m}{\pi} \pm \frac{1}{2}\Big)^{\frac{1}{m}} = 1$$

and the Sandwich Lemma show that

$$\lim_{m\to\infty} q_m^{\frac{1}{m}} = 1\,.$$

Passing to the limit as $m \to \infty$ in (0.4), we conclude that $\lim\limits_{m\to\infty} \sqrt[m]{|\sin m|} = 1$. This shows that

$$\liminf_{m\to\infty} \sqrt[m]{|\sin m|} = \limsup_{m\to\infty} \sqrt[m]{|\sin m|} = 1\,.$$

3. Let $x_m = \big(1 + \frac{1}{m}\big)\sin\frac{m\pi}{6}$. Since $\lim\limits_{m\to\infty}\big(1 + \frac{1}{m}\big) = 1 > 0$ and there are seven cluster points, $\big\{\pm 1, \pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}, 0\big\}$, of the sequence $\big\{\sin\frac{m\pi}{6}\big\}_{m=1}^{\infty}$, we expect that

$$\limsup_{m\to\infty}\Big(1 + \frac{1}{m}\Big)\sin\frac{m\pi}{6} = 1 \qquad \text{and} \qquad \liminf_{m\to\infty}\Big(1 + \frac{1}{m}\Big)\sin\frac{m\pi}{6} = -1\,.$$

To see that our expectation is in fact true, we let $\varepsilon > 0$ be given and observe that

$$\#\big\{m \in \mathbb{N} \,\big|\, x_m > 1 + \varepsilon\big\} \leqslant \Big[\frac{1}{\varepsilon}\Big] + 1 < \infty$$

while the set $\big\{m \in \mathbb{N} \,\big|\, x_m > 1 + \varepsilon\big\} \supseteq \big\{12k + 3 \,\big|\, k \in \mathbb{N}\big\}$ so that

$$\#\big\{m \in \mathbb{N} \,\big|\, x_m > 1 + \varepsilon\big\} = \infty\,.$$

Therefore, Proposition 1.98 shows that 1 is the limit superior of $\{x_m\}_{m=1}^{\infty}$. Similarly, -1 is the limit inferior of $\{x_m\}_{m=1}^{\infty}$. □