Exercise Problem Sets 12

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Problem 1. Show that every polynomial is a tempered distribution.

Proof. Since every polynomial is a linear combination of monomials, it suffices to show that x^α, where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index so that $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. If $\phi \in \mathscr{S}(\mathbb{R}^n)$, then

$$\begin{aligned}\big|\langle x^\alpha, \phi(x)\rangle\big| &\leqslant \int_{\mathbb{R}^n} |x|^{|\alpha|}\big|\phi(x)\big|\,dx \leqslant \int_{\mathbb{R}^n} |x|^{|\alpha|}\langle x\rangle^{-|\alpha|-n-1}\langle x\rangle^{|\alpha|+n+1}\big|\phi(x)\big|\,dx \\ &\leqslant \Big(\int_{\mathbb{R}^n}\langle x\rangle^{-n-1}\,dx\Big)p_{|\alpha|+n+1}(\phi) \leqslant \pi p_{|\alpha|+n+1}(\phi)\,.\end{aligned}$$

Therefore,

$$\big|\langle x^\alpha, \phi(x)\rangle\big| \leqslant \pi p_k(\phi) \qquad \forall\, k \geqslant |\alpha| + n + 1$$

which shows that the function $y = x^\alpha$ is a tempered distribution. □

Problem 2. Let $\{\eta_\epsilon\}_{\epsilon>0}$ is the standard mollifiers, and $\phi \in \mathscr{S}(\mathbb{R}^n)$. Show that $\{\eta_\epsilon * \phi\}_{\epsilon>0}$ converges to ϕ in $\mathscr{S}(\mathbb{R}^n)$.

Proof. Since $D(\eta_\epsilon * \phi) = \eta_\epsilon * D\phi$ and the derivative of a Schwartz function is also a Schwartz function, W.L.O.G. it suffices to show that

$$\lim_{\epsilon\to 0^+}\sup_{x\in\mathbb{R}^n}\langle x\rangle^k\big|(\eta_\epsilon * \phi)(x) - \phi(x)\big| = 0 \qquad \forall\, k \in \mathbb{N}\,.$$

Let $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$ be given. Then

$$\begin{aligned}\big|(\eta_\epsilon * \phi)(x) - \phi(x)\big| &= \Big|\int_{\mathbb{R}^n}\eta_\epsilon(y)\big[\phi(x-y) - \phi(x)\big]\,dy\Big| \\ &= \Big|\int_{B(0,\epsilon)}\eta_\epsilon(y)\Big(\int_0^1\frac{d}{dt}\phi(x-ty)\,dt\Big)\,dy\Big| = \Big|\int_{B(0,\epsilon)}\eta_\epsilon(y)\Big(\int_0^1(\nabla\phi)(x-ty)\cdot y\,dt\Big)\,dy\Big| \\ &\leqslant \epsilon\int_{B(0,\epsilon)}\eta_\epsilon(y)\Big(\int_0^1\big|(\nabla\phi)(x-ty)\big|\,dt\Big)\,dy\,.\end{aligned}$$

By the fact that

$$\langle x\rangle^k \leqslant C_k\big(\langle x-ty\rangle^k + \langle y\rangle^k\big) \quad \forall\, t \in [0,1]$$

for some constant C_k (in fact, C_k can be chosen as $2^k - 1$), we find that

$$\begin{aligned}&\langle x\rangle^k\big|(\eta_\epsilon * \phi)(x) - \phi(x)\big| \\ &\qquad\leqslant C_k\epsilon\int_{B(0,\epsilon)}\eta_\epsilon(y)\Big(\int_0^1\big(\langle x-ty\rangle^k + \langle y\rangle^k\big)\big|(\nabla\phi)(x-ty)\big|\,dt\Big)\,dy \\ &\qquad\leqslant C_k\epsilon\int_{B(0,\epsilon)}\eta_\epsilon(y)\Big(\int_0^1\Big(\sup_{x\in\mathbb{R}^n}\langle x\rangle^k\big|(D\phi)(x)\big| + \langle\epsilon\rangle^k\sup_{x\in\mathbb{R}^n}\big|(D\phi)(x)\big|\Big)\,dt\Big)\,dy \\ &\qquad\leqslant C_k\epsilon\Big(\sup_{x\in\mathbb{R}^n}\langle x\rangle^k\big|(D\phi)(x)\big| + \langle\epsilon\rangle^k\sup_{x\in\mathbb{R}^n}\big|(D\phi)(x)\big|\Big)\,dt\Big)\,.\end{aligned}$$

Therefore,

$$\lim_{\epsilon\to 0}\sup_{x\in\mathbb{R}^n}\langle x\rangle^k\big|(\eta_\epsilon * \phi)(x)-\phi(x)\big| = 0$$

which shows that $\{\eta_\epsilon * \phi\}_{\epsilon>0}$ converges to ϕ in $\mathscr{S}(\mathbb{R}^n)$. □

Problem 3. In this problem we consider the concept of the convergence of sequence of tempered distribution given by the following

Definition 0.1 (Convergence in $\mathscr{S}(\mathbb{R}^n)'$). A sequence of distributions $T_n \in \mathscr{S}(\mathbb{R}^n)'$ is said to converge to $T \in \mathscr{S}(\mathbb{R}^n)'$ *in the sense of distribution*, or *in the distributional sense*, if $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$ as $n\to\infty$ for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

Complete the following.

1. Show that if $T \in \mathscr{S}(\mathbb{R}^n)'$ and $\{\phi_n\}_{n=1}^\infty \subseteq \mathscr{S}(\mathbb{R}^n)$ is a sequence which converges to ϕ in $\mathscr{S}(\mathbb{R}^n)$, then $\lim\limits_{n\to\infty}\langle T, \phi_n\rangle = \langle T, \phi\rangle$.
2. Given the definition above, show that if $T \in \mathscr{S}(\mathbb{R}^n)'$, then $\{\eta_\epsilon * T\}_{\epsilon>0}$ converges to T in the sense of distribution, where $\{\eta_\epsilon\}_{\epsilon>0}$ is the standard mollifiers.

Proof. 1. By the definition of the tempered distribution, there exists $N > 0$ such that for all $k \geqslant N$ there exists C_k such that

$$\big|\langle T, u\rangle\big| \leqslant C_k p_k(u) \qquad \forall\, u \in \mathscr{S}(\mathbb{R}^n)\,.$$

Since $\{\phi_n\}_{n=1}^\infty$ converges to ϕ in $\mathscr{S}(\mathbb{R}^n)$, we have $\lim\limits_{n\to\infty} p_k(\phi_n - \phi) = 0$ for $k \gg 1$. Therefore,

$$\big|\langle T, \phi_n\rangle - \langle T, \phi\rangle\big| \leqslant C_k p_k(\phi_n - \phi) \qquad \forall\, k \geqslant N$$

which implies that $\lim\limits_{n\to\infty}\big|\langle T, \phi_n\rangle - \langle T, \phi\rangle\big| = 0$.

2. By Problem 2, for each $\phi \in \mathscr{S}(\mathbb{R}^n)$ the sequence $\{\eta_\epsilon\}_{\epsilon>0}$ converges to ϕ in $\mathscr{S}(\mathbb{R}^n)$. Therefore, for each $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\lim_{\epsilon\to 0}\langle T * \eta_\epsilon, \phi\rangle = \lim_{\epsilon\to 0}\langle T, \widetilde{\eta}_\epsilon * \phi\rangle = \lim_{\epsilon\to 0}\langle T, \eta_\epsilon * \phi\rangle = 0$$

which shows that $\{\eta_\epsilon * T\}_{\epsilon>0}$ converges to T in the sense of distribution. □

Problem 4. In this problem we discuss the derivative of tempered distributions. Complete the following.

1. Show that

$$\Big\langle \frac{\partial f}{\partial x_j}, g\Big\rangle = -\Big\langle f, \frac{\partial g}{\partial x_j}\Big\rangle \qquad \forall\, f, g \in \mathscr{S}(\mathbb{R}^n)\,.$$

This leads to the definition of the derivatives of tempered distributions: Let $T \in \mathscr{S}(\mathbb{R}^n)'$ be a tempered distribution. The partial derivative of T w.r.t. x_j, denoted by $\dfrac{\partial T}{\partial x_j}$, is a tempered distribution defined by

$$\Big\langle \frac{\partial T}{\partial x_j}, \phi \Big\rangle = -\Big\langle T, \frac{\partial \phi}{\partial x_j} \Big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

Verify that $\dfrac{\partial T}{\partial x_j}$ is indeed a tempered distribution; that is, show that there exists a sequence $\{C_k\}_{k=1}^\infty$ such that

$$\Big|\Big\langle \frac{\partial T}{\partial x_j}, \phi \Big\rangle\Big| \leqslant C_k p_k(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n) \text{ and } k \gg 1\,.$$

2. Show that for $1 \leqslant j \leqslant n$,

$$\mathscr{F}_x\Big[\frac{\partial T}{\partial x_j}\Big](\xi) = i\xi_j \widehat{T}(\xi) \qquad \text{and} \qquad \frac{\partial}{\partial x_j}\widehat{T}(\xi) = -i\mathscr{F}_x[xT(x)](\xi)$$

or to be more precise,

$$\Big\langle \widehat{\frac{\partial T}{\partial x_j}}, \phi \Big\rangle = \big\langle \widehat{T}(\xi), i\xi_j \phi(\xi) \big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)$$

and

$$\Big\langle \frac{\partial}{\partial \xi_j}\widehat{T}(\xi), \phi(\xi) \Big\rangle = \big\langle T(x), -ix\widehat{\phi}(x) \big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,.$$

In other words, the Fourier transform of derivatives of tempered distributions still obeys Lemma 9.9 and 9.11 in the lectute note.

Proof. 1. Let $f, g \in \mathscr{S}(\mathbb{R}^n)$. Then with $\widehat{x}_j$ denoting $(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$,

$$\Big\langle \frac{\partial f}{\partial x_j}, g \Big\rangle = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) g(x)\, dx = \int_{\mathbb{R}^{n-1}} \lim_{R\to\infty} \Big(\int_{-R}^{R} \frac{\partial f}{\partial x_j}(x) g(x)\, dx_j \Big) d\widehat{x}_j\,.$$

Integrating by parts,

$$\begin{aligned}\lim_{R\to\infty} \int_{-R}^{R} \frac{\partial f}{\partial x_j}(x) g(x)\, dx_j &= \lim_{R\to\infty} \Big[f(x)g(x)\Big|_{x_j=-R}^{x_j=R} - \int_{-R}^{R} f(x) \frac{\partial g}{\partial x_j}(x)\, dx_j \Big] \\ &= -\int_{-\infty}^{\infty} f(x) \frac{\partial g}{\partial x_j}(x)\, dx_j\,;\end{aligned}$$

thus

$$\Big\langle \frac{\partial f}{\partial x_j}, g \Big\rangle = -\int_{\mathbb{R}^n} f(x) \frac{\partial g}{\partial x_j}(x)\, dx = -\Big\langle f, \frac{\partial g}{\partial x_j} \Big\rangle.$$

Suppose that $T \in \mathscr{S}(\mathbb{R}^n)'$. Then there exists $N > 0$ and a sequence $\{C_k\}_{k=1}^\infty$ such that

$$\big|\langle T, \phi \rangle\big| \leqslant C_k p_k(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n) \text{ and } k \geqslant N\,.$$

Therefore, if $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $k \geqslant N$, by the fact that

$$p_k\Big(\frac{\partial \phi}{\partial x_j}\Big) = \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k} \langle x \rangle^k \Big| D^\alpha \frac{\partial \phi}{\partial x_j}(x) \Big| \leqslant \sup_{x\in\mathbb{R}^n, |\alpha|\leqslant k+1} \langle x \rangle^{k+1} \big| D^\alpha \phi(x) \big| = p_{k+1}(\phi)$$

we find that

$$\Big|\Big\langle \frac{\partial T}{\partial x_j}, \phi \Big\rangle\Big| = \Big|\Big\langle T, \frac{\partial \phi}{\partial x_j} \Big\rangle\Big| \leqslant C_k p_k\Big(\frac{\partial \phi}{\partial x_j}\Big) \leqslant C_k p_{k+1}(\phi)\,;$$

thus $\dfrac{\partial T}{\partial x_j}$ is a tempered distribution.

2. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\Big\langle \widehat{\frac{\partial T}{\partial x_j}}, \phi \Big\rangle = \Big\langle \frac{\partial T}{\partial x_j}, \widehat{\phi} \Big\rangle = -\big\langle T, \frac{\partial}{\partial x_j}\widehat{\phi}(x) \big\rangle .$$

By Lemma 9.9 in the lecture note, $\dfrac{\partial}{\partial x_j}\widehat{\phi}(x) = \mathscr{F}_\xi \Big[\frac{1}{i}\xi_j \phi(\xi)\Big](x)$; thus

$$\Big\langle \widehat{\frac{\partial T}{\partial x_j}}, \phi \Big\rangle = -\Big\langle T(x), \mathscr{F}_\xi \Big[\frac{1}{i}\xi_j \phi(\xi)\Big](x) \Big\rangle = \big\langle \widehat{T}(\xi), i\xi_j \phi(\xi) \big\rangle . \qquad \square$$

Problem 5. Complete the following.

1. Show that if $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, then the distribution $T * \phi$ is indeed a smooth function.
2. Show that if $T' = 0$ in $\mathscr{S}(\mathbb{R})'$, where the derivative of tempered distribution is given by Problem 4, then T is a constant; that is, there exists $C \in \mathbb{R}$ such that

$$\langle T, \phi \rangle = \langle C, \phi \rangle = C \int_{\mathbb{R}} \phi(x)\, dx \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}) .$$

Proof. 1. First we claim that if $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, then $\dfrac{\partial}{\partial x_j}(T * \phi) = T * \dfrac{\partial \phi}{\partial x_j}$: for $\psi \in \mathscr{S}(\mathbb{R}^n)$, by the fact that

$$\frac{\partial}{\partial x_j}(\phi * \psi)(x) = \Big(\frac{\partial \phi}{\partial x_j} * \psi\Big)(x) = \Big(\phi * \frac{\partial \psi}{\partial x_j}\Big)(x) ,$$

we have

$$\begin{aligned} \Big\langle \frac{\partial}{\partial x_j}(T * \phi), \psi \Big\rangle &= -\Big\langle T * \phi, \frac{\partial \psi}{\partial x_j} \Big\rangle = -\Big\langle T, \widetilde{\phi} * \frac{\partial \psi}{\partial x_j} \Big\rangle = -\langle T, \frac{\partial \widetilde{\phi}}{\partial x_j} * \psi \rangle \\ &= \langle T, \widetilde{\frac{\partial \phi}{\partial x_j}} * \psi \rangle = \Big\langle T * \frac{\partial \phi}{\partial x_j}, \psi \Big\rangle . \end{aligned}$$

Therefore, it suffices to show that $T * \phi$ is a function (for all $\phi \in \mathscr{S}(\mathbb{R}^n)$).

For $f, g, h \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned} \langle f * g, h \rangle &= \int_{\mathbb{R}^n} (f * g)(x) h(x)\, dx = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f(y) g(x-y)\, dy \Big) h(x)\, dx \\ &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} f(y) (\tau_x \widetilde{g})(y)\, dy \Big) h(x)\, dx = \int_{\mathbb{R}^n} \langle f, \tau_x \widetilde{g} \rangle h(x)\, dx . \end{aligned}$$

Therefore, it motivates the definition

$$(T * \phi)(x) = \big\langle T, \tau_x \widetilde{\phi} \big\rangle$$

whenever the right-hand side exists. This is the case if $\phi \in \mathscr{S}(\mathbb{R}^n)$; thus $T * \phi$ is identical to the function $g(x) = \big\langle T, \tau_x \widetilde{\phi} \big\rangle$.

2. Since $T' = 0$, we find that $T' * \phi = 0$ for all $\phi \in \mathscr{S}(\mathbb{R})$. By (1), $T' * \phi = (T * \phi)'$ is a smooth function, so for each $\phi \in \mathscr{S}(\mathbb{R})$ there exists a constant $C = C(\phi)$ such that

$$T * \phi = C(\phi)\,.$$

In particular, letting $\phi = \eta_\epsilon$ we find that $T * \eta_\epsilon = C_\epsilon$ for some constant C_ϵ. Therefore, for each $\phi \in \mathscr{S}(\mathbb{R})$,

$$\langle T * \eta_\epsilon, \phi \rangle = C_\epsilon \int_{\mathbb{R}} \phi(x)\, dx\,.$$

On the other hand,

$$\langle T * \eta_\epsilon, \phi \rangle = \langle T, \widetilde{\eta_\epsilon} * \phi \rangle = \langle T, \eta_\epsilon * \phi \rangle$$

which, using results from Problem 2 and 3, shows that

$$\lim_{\epsilon \to 0^+} \langle T * \eta_\epsilon, \phi \rangle = \langle T, \phi \rangle\,.$$

This implies that

$$\lim_{\epsilon \to 0^+} C_\epsilon \int_{\mathbb{R}} \phi(x)\, dx = \langle T, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$

Therefore, $\lim\limits_{\epsilon \to 0^+} C_\epsilon = C$ exists, and we conclude that

$$C \int_{\mathbb{R}} \phi(x)\, dx = \langle T, \phi \rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$ □

Problem 6. Let sgn : $\mathbb{R} \to \mathbb{R}$ be the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0\,, \\ -1 & \text{if } x < 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then clearly sgn is a tempered distribution since

$$\big|\langle \operatorname{sgn}, \phi \rangle\big| \leqslant \|\phi\|_{L^1(\mathbb{R})} \leqslant \pi p_2(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$

Show that $\dfrac{d}{dx}\operatorname{sgn}(x) = 2\delta$ in $\mathscr{S}(\mathbb{R})'$, where the derivative of tempered distributions is defined in Problem 4 and δ is the Dirac delta function.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R})$. Then by definition of the derivatives of tempered distributions,

$$\begin{aligned} \Big\langle \frac{d}{dx}\operatorname{sgn}(x), \phi(x) \Big\rangle &= -\big\langle \operatorname{sgn}(x), \phi'(x) \big\rangle = -\int_{-\infty}^{\infty} \operatorname{sgn}(x)\phi'(x)\, dx \\ &= -\int_0^\infty \phi'(x)\, dx + \int_{-\infty}^0 \phi'(x)\, dx \\ &= -\phi(x)\Big|_{x=0}^{x=\infty} + \phi(x)\Big|_{x=-\infty}^{x=0} = 2\phi(0) = \langle 2\delta, \phi \rangle \end{aligned}$$

which shows that $\dfrac{d}{dx}\operatorname{sgn}(x) = 2\delta$ in $\mathscr{S}(\mathbb{R})'$. □

Problem 7. Compute the Fourier transform of the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = |x|^\alpha$, where $-n < \alpha < 0$, by the following procedure.

1. Show that $f \notin L^1(\mathbb{R}^n)$.

2. Recall that the Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ defined by $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$. Show that

$$|x|^\alpha = \frac{1}{\Gamma(-\frac{\alpha}{2})}\int_0^\infty s^{-\frac{\alpha}{2}-1}e^{-s|x|^2}\,ds \qquad \forall\, x \neq 0\,.$$

3. Find that Fourier transform of f.

4. Find the Fourier transform of the function $g : \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = x_1|x|^\alpha$, where x_1 is the first component of x and $-n-2 < \alpha < -2$.

Hint: 3. Compute $\langle |x|^\alpha, \widehat{\phi}(x)\rangle$ by applying Fubini's Theorem several times.

4. Note that $g(x) = \dfrac{1}{\alpha+2}\dfrac{\partial}{\partial x_1}|x|^{\alpha+2}$ so that you can apply the results above. See Problem 4 for the Fourier transform of derivatives of tempered distributions.

Proof. 1. By the change of variables formula,

$$\int_{\mathbb{R}^n} |x|^\alpha\,dx = \int_{\mathbb{S}^{n-1}}\int_0^\infty r^\alpha r^{n-1} dr dS = \omega_{n-1}\int_0^\infty r^{\alpha+n-1}\,dr = \infty\,.$$

Therefore, $f \notin L^1(\mathbb{R}^n)$.

2. By the substitution of variable $s|x|^2 = t$ (for $x \neq 0$),

$$\int_0^\infty s^{-\frac{\alpha}{2}-1}e^{-s|x|^2}\,ds = \int_0^\infty |x|^{\alpha+2}t^{-\frac{\alpha}{2}-1}e^{-t}|x|^{-2}\,dt = |x|^\alpha\int_0^\infty t^{-\frac{\alpha}{2}-1}e^{-t}\,dt = |x|^\alpha\Gamma\big(-\frac{\alpha}{2}\big)\,.$$

Therefore, $|x|^\alpha = \dfrac{1}{\Gamma\big(-\frac{\alpha}{2}\big)}\displaystyle\int_0^\infty s^{-\frac{\alpha}{2}-1}e^{-s|x|^2}\,ds$.

3. For a given Schwartz function $\phi \in \mathscr{S}(\mathbb{R}^n)$, define $g(x,s) = s^{-\frac{\alpha}{2}-1}e^{-s|x|^2}\widehat{\phi}(x)$ and $h(\xi,s) = s^{-\frac{n}{2}-\frac{\alpha}{2}-1}e^{-\frac{|\xi|^2}{4s}}\phi(\xi)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n\times(0,\infty)} \big|g(x,s)\big|\,d(x,s) &= \int_{\mathbb{R}^n}\Big(\int_0^\infty s^{-\frac{\alpha}{2}-1}e^{-s|x|^2}\big|\widehat{\phi}(x)\big|\,ds\Big)\,dx \\ &= \int_{\mathbb{R}^n}|x|^\alpha\big|\widehat{\phi}(x)\big|\,dx = \int_{\mathbb{S}^{n-1}}\Big(\int_0^\infty r^{n+\alpha-1}\big|\widehat{\phi}(r\omega)\big|\,dr\Big)dS\end{aligned}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n\times(0,\infty)} \big|h(\xi,s)\big|\, d(\xi,s) \\
&\quad = \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}} \big|\phi(\xi)\big|\, ds \Big)\, d\xi = \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}}\, ds \Big) \big|\phi(\xi)\big|\, d\xi \\
&\quad = \int_{\mathbb{R}^n} \Big(\int_0^\infty (4t)^{\frac{n}{2}+\frac{\alpha}{2}+1} |\xi|^{-n-\alpha-2} e^{-t} \frac{|\xi|^2}{4t^2}\, dt \Big) \big|\phi(\xi)\big|\, d\xi \\
&\quad = 2^{n+\alpha} \int_{\mathbb{R}^n} \Big(\int_0^\infty t^{\frac{n+\alpha}{2}-1} e^{-t}\, dt \Big) |\xi|^{-n-\alpha} \big|\phi(\xi)\big|\, d\xi \\
&\quad = 2^{n+\alpha} \Gamma\big(\frac{n+\alpha}{2}\big) \int_{\mathbb{R}^n} |\xi|^{-n-\alpha} \big|\phi(\xi)\big|\, d\xi\,.
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_0^\infty r^{n+\alpha-1} \big|\widehat{\phi}(r\omega)\big|\, dr &\leqslant \|\widehat{\phi}\|_\infty \int_0^1 r^{n+\alpha-1} dr + \sup_{x\in\mathbb{R}^n} \big(|x|^n \big|\widehat{\phi}(x)\big|\big) \int_1^\infty r^{\alpha-1} dr \\
&\leqslant \frac{\|\phi\|_{L^1(\Omega)}}{n+\alpha} + \frac{1}{-\alpha} \sup_{x\in\mathbb{R}^n} \big(|x|^n \big|\widehat{\phi}(x)\big|\big) < \infty
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^n} |\xi|^{-n-\alpha} \big|\phi(\xi)\big|\, d\xi \leqslant \int_{\mathbb{R}^n} \langle\xi\rangle^{-n-1} \langle\xi\rangle^{1-\alpha} \big|\phi(\xi)\big|\, d\xi \leqslant \big\|\langle\xi\rangle\big\|_{L^1(\mathbb{R}^n)} \sup_{\xi\in\mathbb{R}^n} \langle\xi\rangle^{1-\alpha} \big|\phi(\xi)\big| < \infty\,,
$$

we find that g and h are integrable on $\mathbb{R}^n \times (0,\infty)$. By the definition of the Fourier transform of tempered distributions,

$$
\langle |x|^\alpha, \widehat{\phi} \rangle = \int_{\mathbb{R}^n} |x|^\alpha \widehat{\phi}(x)\, dx = \frac{1}{\Gamma\big(-\frac{\alpha}{2}\big)} \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\, ds \Big) \widehat{\phi}(x)\, dx
$$

and the Fubini Theorem (which can be applied since g is integrable on $\mathbb{R}^n \times (0,\infty)$) implies that

$$
\begin{aligned}
&\Gamma\big(-\frac{\alpha}{2}\big) \langle |x|^\alpha, \widehat{\phi} \rangle \\
&\quad = \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{\alpha}{2}-1} e^{-s|x|^2}\, ds \Big) \widehat{\phi}(x)\, dx = \int_0^\infty \Big(\int_{\mathbb{R}^n} s^{-\frac{\alpha}{2}-1} e^{-s|x|^2} \widehat{\phi}(x)\, dx \Big)\, ds \\
&\quad = \int_0^\infty s^{-\frac{\alpha}{2}-1} \langle e^{-s|x|^2}, \widehat{\phi}(x) \rangle\, ds = \int_0^\infty s^{-\frac{\alpha}{2}-1} \langle \mathscr{F}_x[e^{-s|x|^2}](\xi), \phi(\xi) \rangle\, ds \\
&\quad = \int_0^\infty s^{-\frac{\alpha}{2}-1} \Big(\int_{\mathbb{R}^n} (2s)^{-\frac{n}{2}} e^{-\frac{|\xi|^2}{4s}} \phi(\xi)\, d\xi \Big)\, ds \\
&\quad = 2^{-\frac{n}{2}} \int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} \Big(\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{4s}} \phi(\xi)\, d\xi \Big)\, ds\,.
\end{aligned}
$$

By the integrability of h on $\mathbb{R}^n \times (0,\infty)$, we can apply the Fubini Theorem to obtain that

$$
\begin{aligned}
\Gamma\big(-\frac{\alpha}{2}\big) \langle |x|^\alpha, \widehat{\phi} \rangle &= 2^{-\frac{n}{2}} \int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} \Big(\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{4s}} \phi(\xi)\, d\xi \Big)\, ds \\
&= 2^{-\frac{n}{2}} \int_{\mathbb{R}^n} \Big(\int_0^\infty s^{-\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\frac{|\xi|^2}{4s}}\, ds \Big) \phi(\xi)\, d\xi = 2^{\frac{n}{2}+\alpha} \Gamma\big(\frac{n+\alpha}{2}\big) \int_{\mathbb{R}^n} |\xi|^{-n-\alpha} \phi(\xi)\, d\xi \\
&= 2^{\frac{n}{2}+\alpha} \Gamma\big(\frac{n+\alpha}{2}\big) \langle |\xi|^{-n-\alpha}, \phi(\xi) \rangle\,.
\end{aligned}
$$

Therefore, $\mathscr{F}_x[|x|^\alpha](\xi) = \dfrac{\Gamma\big(\frac{n+\alpha}{2}\big)}{\Gamma\big(-\frac{\alpha}{2}\big)} 2^{\frac{n}{2}+\alpha}|\xi|^{-\alpha-n}$.

4. First we shown that if the partial derivative of tempered distributions is given by

$$\Big\langle \frac{\partial T}{\partial x_j}, \phi \Big\rangle = -\Big\langle T, \frac{\partial \phi}{\partial x_j} \Big\rangle \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}^n)\,,$$

then $\mathscr{F}_x\big[\frac{\partial T}{\partial x_j}\big](\xi) = i\xi_j \widehat{T}(\xi)$. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then Lemma 9.11 implies that

$$\frac{\partial}{\partial x_j}\widehat{\phi}(x) = -i\mathscr{F}_\xi\big[\xi_j \phi(\xi)\big](x)\,;$$

thus

$$\begin{aligned} \Big\langle \mathscr{F}\Big(\frac{\partial T}{\partial x_j}\Big), \phi \Big\rangle &= \Big\langle \frac{\partial T}{\partial x_j}, \widehat{\phi} \Big\rangle = -\Big\langle T, \frac{\partial}{\partial x_j}\widehat{\phi}(x) \Big\rangle = -\big\langle T, -i\mathscr{F}_\xi[\xi_j\phi(\xi)](x) \big\rangle \\ &= i\big\langle \widehat{T}, \xi_j\phi(\xi) \big\rangle = \big\langle i\xi_j \widehat{T}(\xi), \phi(\xi) \big\rangle\,. \end{aligned}$$

Therefore, $\mathscr{F}\big[\frac{\partial T}{\partial x_j}\big](\xi) = i\xi_j \widehat{T}(\xi)$.

Now, since $g(x) = \dfrac{1}{\alpha+2}\dfrac{\partial}{\partial x_1}|x|^{\alpha+2}$, by the fact that $|x|^{\alpha+2}$ is a tempered distribution for $-n < \alpha+2 < 0$, we conclude that if $-n-2 < \alpha < -2$, we have

$$\widehat{g}(\xi) = \frac{1}{\alpha+2} i\xi_1 \mathscr{F}_x\big[|x|^{\alpha+2}\big](\xi) = i\frac{\Gamma\big(\frac{n+\alpha+2}{2}\big)}{\Gamma\big(-\frac{\alpha+2}{2}\big)} \frac{2^{\frac{n}{2}+\alpha+2}}{\alpha+2} \xi_1 |\xi|^{-\alpha-n-2}\,. \qquad \square$$

Problem 8. Let $f \in L^1(\mathbb{R})$. Show that the function $y = \displaystyle\int_{-\infty}^x f(t)\,dt$ can be written as the convolution of f and a function $\phi \in L^1_{\text{loc}}(\mathbb{R})$.

Proof. Let ϕ be the characteristic function of the set $(0,\infty)$, or

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0\,, \\ 0 & \text{if } x \leqslant 0\,. \end{cases}$$

Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$, and

$$(\phi * f)(x) = \int_{\mathbb{R}} \phi(x-y) f(y)\,dy = \int_{-\infty}^x f(y)\,dy$$

which is the anti-derivative of f. $\square$

Problem 9. In this problem we use symbolic computation to find the Fourier transform of the function

$$f(x) = \begin{cases} \dfrac{\sin(\omega x)}{x} & \text{if } x \neq 0\,, \\ \omega & \text{if } x = 0\,, \end{cases}$$

without knowing that it is the Fourier transform of the function $y = \sqrt{\frac{\pi}{2}}\chi_{(-\omega,\omega)}(x)$ (where $\chi_{(-\omega,\omega)}$ is the characteristic/indicator function of the set $(-\omega,\omega)$). Complete the following.

1. Note that $f \notin L^1(\mathbb{R})$ but $f \in \mathscr{S}(\mathbb{R}^n)'$. Therefore, $\widehat{f} \in \mathscr{S}(\mathbb{R})$. Find the derivative of $\widehat{f}$, where the derivatives of tempered distributions is given in Problem 4.

2. Suppose that you can use the Fundamental Theorem of Calculus so that

$$\widehat{f}(\xi) - \widehat{f}(0) = \int_0^\xi \widehat{f}\,'(t)\,dt\,.$$

Note that in Problem 7 of Exercise 3 you are asked to show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. Use this fact and treat $\delta_{\pm\omega}$ as the evaluation operation at $\pm\omega$ to find $\widehat{f}(\xi)$ (for $\xi \neq \pm\omega$).

Hint: 1. Recall that we have shown in Example 9.48 that $\mathscr{F}_x[\sin(\omega x)](\xi) = \dfrac{\sqrt{2\pi}}{2i}(\delta_\omega - \delta_{-\omega})$.

Proof. 1. Let $\phi \in \mathscr{S}(\mathbb{R})$. By the definition of the derivative of tempered distributions,

$$\begin{aligned}\langle \widehat{f}\,', \phi\rangle &= -\langle \widehat{f}, \phi'\rangle = -\langle f, \widehat{\phi'}\rangle = -\langle f(x), ix\widehat{\phi}(x)\rangle = -i\langle \sin(\omega x), \widehat{\phi}(x)\rangle \\ &= -i\langle \mathscr{F}_x[\sin(\omega x)](\xi), \phi(\xi)\rangle\,;\end{aligned}$$

thus

$$\widehat{f}\,'(\xi) = -i\mathscr{F}_x[\sin(\omega x)](\xi) = -\sqrt{\frac{\pi}{2}}(\delta_\omega - \delta_{-\omega})\,.$$

2. Note that

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\sin(\omega x)}{x} e^{ix\cdot 0}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy = \sqrt{\frac{\pi}{2}}\,;$$

thus the Fundamental Theorem of Calculus implies that

$$\widehat{f}(\xi) = \widehat{f}(0) + \int_0^\xi \widehat{f}'(t)\,dt = \sqrt{\frac{\pi}{2}}\Big[1 - \int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt\Big]\,.$$

(a) If $\xi < 0$, then

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = -\int_{\mathbb{R}} \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\mathbf{1}_{[\xi,0]}(t)\,dt = \mathbf{1}_{[\xi,0]}(-\omega)\,;$$

thus

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = \begin{cases} 0 & \text{if } -\omega < \xi < 0\,, \\ 1 & \text{if } \xi < -\omega\,. \end{cases}$$

(b) If $\xi > 0$, then

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = \int_{\mathbb{R}} \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\mathbf{1}_{[0,\xi]}(t)\,dt = \mathbf{1}_{[0,\xi]}(\omega)\,;$$

thus

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\,dt = \begin{cases} 0 & \text{if } 0 < \xi < \omega\,, \\ 1 & \text{if } \xi > \omega\,. \end{cases}$$

Therefore,

$$\int_0^\xi \big[\delta_\omega(t) - \delta_{-\omega}(t)\big]\, dt = \begin{cases} 0 & \text{if } -\omega < \xi < \omega\,, \\ 1 & \text{if } |\xi| > \omega\,, \end{cases}$$

which shows that $\widehat{f}(\xi) = \sqrt{\frac{\pi}{2}}\mathbf{1}_{(-\omega,\omega)}(\xi)$. □

Problem 10. Let ω be a positive real number, and $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{\sin(\omega|x|)}{|x|} & \text{if } x \neq 0\,, \\ \omega & \text{if } x = 0\,, \end{cases}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ if $x = (x_1, x_2, x_3)$. In this problem we are concerned with the Fourier transform of f. Complete the following.

1. Show that $f \in \mathscr{S}(\mathbb{R}^3)'$.

2. Show that the Fourier transform of f is given by

$$\langle \widehat{f}, \varphi\rangle = \sqrt{\frac{\pi}{2}}\frac{1}{\omega}\int_{\partial B(0,\omega)} \varphi\, dS$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^3)$, where $\displaystyle\int_{\partial B(0,\omega)} \varphi\, dS$ is the surface integral of φ on the sphere $\partial B(0,\omega)$ defined by

$$\int_{\partial B(0,\omega)} \varphi\, dS \equiv \int_0^\pi \int_0^{2\pi} \varphi(\omega\cos\theta\sin\phi, \omega\sin\theta\sin\phi, \omega\cos\phi)\omega^2\sin\phi\, d\theta d\phi\,.$$

Hint of 2: You can show part 2 through the following procedures:

Step 1: By the definition of the Fourier transform of the tempered distributions,

$$\big\langle \widehat{f}, \varphi\big\rangle = \big\langle f, \widehat{\varphi}\big\rangle = \lim_{m\to\infty}\int_{B(0,m)} f(x)\Big(\frac{1}{\sqrt{2\pi}^3}\int_{\mathbb{R}^3} \varphi(\xi)e^{-ix\cdot\xi}\, d\xi\Big)\, dx$$

and the Fubini Theorem implies that

$$\big\langle \widehat{f}, \varphi\big\rangle = \frac{1}{\sqrt{2\pi}^3}\lim_{m\to\infty}\int_{\mathbb{R}^3}\Big(\int_{B(0,m)} f(x)e^{-ix\cdot\xi}\, dx\Big)\varphi(\xi)d\xi\,.$$

We focus on the inner integral first. Show that for each 3×3 orthonormal matrix O,

$$\int_{B(0,m)} f(x)e^{-ix\cdot\xi}\, dx = \int_{B(0,m)} \frac{\sin(\omega|y|)}{|y|}\, e^{-i(\mathrm{O}^{\mathrm{T}}\xi)\cdot y}\, dy\,.$$

Step 2: For each $\xi \in \mathbb{R}^3$, choose a 3×3 orthonormal matrix O such that $\mathrm{O}^{\mathrm{T}}\xi = (0, 0, |\xi|)$. Using the spherical coordinate $y = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$ to show that

$$\int_{B(0,m)} f(x)e^{-ix\cdot\xi}\, dx = \int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|}\, d\rho$$

so that we conclude that

$$\big\langle \widehat{f}, \varphi\big\rangle = \frac{1}{\sqrt{2\pi}^3}\lim_{m\to\infty}\int_{\mathbb{R}^3}\Big(\int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|}\varphi(\xi)\, d\rho\Big)d\xi\,.$$

Step 3: For each $r > 0$, define $\psi(r)$ as the surface integral of φ on $\partial B(0,r)$; that is,

$$\psi(r) = \int_{\partial B(0,r)} \varphi\, dS \equiv \int_0^\pi \int_0^{2\pi} \varphi(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) r^2 \sin\phi\, d\theta d\phi\,.$$

Using the spherical coordinate $\xi = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ to show that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big(\int_0^\infty \sin(\omega\rho)\sin(r\rho) \frac{2\psi(r)}{r}\, dr \Big) d\rho\,.$$

Step 4: Apply the conclusion in Problem 4 of Exercise 11.

Proof. 1. Since $|f|$ is bounded by ω, by Example 9.38 in the lecture note we immediately obtain that $f \in \mathscr{S}(\mathbb{R}^3)'$.

2. Let $\varphi \in \mathscr{S}(\mathbb{R}^3)$ be a Schwartz function. By the definition of the Fourier transform and the hint,

$$\begin{aligned}
\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle &= \int_{\mathbb{R}^3} f(x) \Big(\frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \varphi(\xi) e^{-ix\cdot\xi}\, d\xi \Big)\, dx \\
&= \lim_{m\to\infty} \int_{B(0,m)} f(x) \Big(\frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \varphi(\xi) e^{-ix\cdot\xi}\, d\xi \Big)\, dx \\
&= \frac{1}{\sqrt{2\pi}^3} \lim_{m\to\infty} \int_{B(0,m)} \Big(\int_{\mathbb{R}^3} f(x)\varphi(\xi) e^{-ix\cdot\xi}\, d\xi \Big)\, dx\,.
\end{aligned}$$

Since the function $g(x,\xi) = f(x)\varphi(\xi)e^{-ix\cdot\xi}$ is integrable on $B(0,m) \times \mathbb{R}^3$, the Fubini Theorem implies that

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}^3} \lim_{m\to\infty} \int_{\mathbb{R}^3} \Big(\int_{B(0,m)} f(x) e^{-ix\cdot\xi}\, dx \Big) \varphi(\xi) d\xi\,.$$

Note that if O is a 3×3 orthonormal matrix, then $|\mathrm{O}^{\mathrm{T}} x| = |x|$ for all $x \in \mathbb{R}^3$; thus for any orthonormal matrix O and $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
\int_{B(0,m)} f(x) e^{-ix\cdot\xi}\, dx &= \int_{B(0,m)} \frac{\sin(\omega|\mathrm{O}^{\mathrm{T}} x|)}{|\mathrm{O}^{\mathrm{T}} x|} e^{-ix\cdot\xi}\, dx \\
(\mathrm{O}^{\mathrm{T}} x = y) &= \int_{B(0,m)} \frac{\sin(\omega|y|)}{|y|} e^{-i(\mathrm{O}y)\cdot\xi}\, dy = \int_{B(0,m)} \frac{\sin(\omega|y|)}{|y|} e^{-i(\mathrm{O}^{\mathrm{T}}\xi)\cdot y}\, dy\,.
\end{aligned}$$

Now, for each $\xi \in \mathbb{R}^n$, choose a 3×3 orthonormal matrix O such that $\mathrm{O}^{\mathrm{T}}\xi = (0,0,|\xi|)$. Using the spherical coordinate $y = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$, we obtain that

$$\begin{aligned}
\int_{B(0,m)} f(x) e^{-ix\cdot\xi}\, dx &= \int_0^m \int_0^\pi \int_0^{2\pi} \frac{\sin(\omega\rho)}{\rho} e^{-i|\xi|\rho\cos\phi} \rho^2 \sin\phi\, d\theta d\phi d\rho \\
&= \int_0^m \sin(\omega\rho) \frac{e^{-i|\xi|\rho\cos\phi}}{i|\xi|}\Big|_{\phi=0}^{\phi=\pi} d\rho = \int_0^m \sin(\omega\rho) \frac{e^{i|\xi|\rho} - e^{-i|\xi|\rho}}{i|\xi|} d\rho \\
&= \int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|}\, d\rho
\end{aligned}$$

so we have

$$\langle \widehat{f}, \varphi \rangle = \frac{1}{\sqrt{2\pi}^3} \lim_{m\to\infty} \int_{\mathbb{R}^3} \Big(\int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|} \varphi(\xi)\, d\rho \Big) d\xi \,.$$

For each $r > 0$, define

$$\psi(r) \equiv \int_{\partial B(0,r)} \phi(x)\, dS = \int_0^\pi \int_0^{2\pi} \phi(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) r^2 \sin\phi\, d\theta d\phi \,.$$

Using the spherical coordinate $\xi = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$, by the Fubini theorem we find that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Big(\int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|} \varphi(\xi)\, d\rho \Big) d\xi \\
&= \int_0^\infty \int_0^\pi \int_0^{2\pi} \Big(\int_0^m \frac{2\sin(\omega\rho)\sin(r\rho)}{r} \varphi(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)\, d\rho \Big) r^2 \sin\phi\, d\theta d\phi dr \\
&= \int_0^\infty \int_0^m \frac{2\sin(\omega\rho)\sin(r\rho)}{r} \Big(\int_0^\pi \int_0^{2\pi} \varphi(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) r^2 \sin\phi\, d\theta d\phi \Big) d\rho dr \\
&= \int_0^\infty \int_0^m \sin(\omega\rho)\sin(r\rho) \frac{2\psi(r)}{r}\, d\rho dr = \int_0^m \Big(\int_0^\infty \sin(\omega\rho)\sin(r\rho) \frac{2\psi(r)}{r}\, dr \Big) d\rho \,;
\end{aligned}$$

thus

$$\begin{aligned}
\langle \widehat{f}, \varphi \rangle &= \frac{1}{\sqrt{2\pi}} \lim_{m\to\infty} \int_{\mathbb{R}^3} \Big(\int_0^m \frac{2\sin(\omega\rho)\sin(|\xi|\rho)}{|\xi|} \varphi(\xi)\, d\rho \Big) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \lim_{m\to\infty} \int_0^m \Big(\int_0^\infty \sin(\omega\rho)\sin(r\rho) \frac{2\psi(r)}{r}\, dr \Big) d\rho \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big(\int_0^\infty \sin(\omega\rho)\sin(r\rho) \frac{2\psi(r)}{r}\, dr \Big) d\rho \,.
\end{aligned}$$

By Problem 4 of Exercise 11,

$$\frac{2}{\pi} \int_0^\infty \Big(\int_0^\infty \sin(\omega\rho)\sin(r\rho) \frac{\psi(r)}{r}\, dr \Big) d\rho = \frac{\psi(\omega)}{\omega} \,;$$

thus

$$\langle \widehat{f}, \varphi \rangle = \sqrt{\frac{\pi}{2}} \frac{\psi(\omega)}{\omega} = \sqrt{\frac{\pi}{2}} \frac{1}{\omega} \int_{\partial B(0,\omega)} \phi(x)\, dS \,.$$

□

Problem 11. 1. Show that the function $R : \mathbb{R} \to \mathbb{R}$ given by

$$\mathrm{R}(x) = \begin{cases} x & \text{if } x \geqslant 0 \,, \\ 0 & \text{otherwise} \,, \end{cases}$$

is a tempered distribution.

2. Let T be a generalized function defined by

$$\langle T, \phi \rangle = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \frac{\phi(x)}{x}\, dx = \lim_{\epsilon\to 0^+} \Big(\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty \Big) \frac{\phi(x)}{x}\, dx \qquad \forall\, \phi \in \mathscr{C}_c^\infty(\mathbb{R}) \,.$$

Show that $T \in \mathscr{S}(\mathbb{R})'$.

3. Let H be the Heaviside function given by

$$H(x) = \begin{cases} 0 & \text{if } x \leqslant 0\,, \\ 1 & \text{if } x > 0\,. \end{cases}$$

Show that $\widehat{H} = \dfrac{-i}{\sqrt{2\pi}} T + \sqrt{\dfrac{\pi}{2}}\,\delta$, here δ is the Dirac delta function.

Hint: 3. Let $G(x) = \exp\left(-\dfrac{x^2}{2}\right)$. For each $\phi \in \mathscr{S}(\mathbb{R})$, define $\psi = \phi - \phi(0)G$ (which belongs to $\mathscr{S}(\mathbb{R})$). Use the identity

$$\langle \widehat{H}, \phi\rangle = \langle H, \widehat{\psi}\rangle - \phi(0)\langle H, \widehat{G}\rangle$$

to make the conclusion.

Proof. 1. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} \big|\langle R, \phi\rangle\big| &= \Big|\int_0^\infty x\phi(x)\,dx\Big| \leqslant \Big(\int_0^\infty |x|\langle x\rangle^{-3}\,dx\Big) \sup_{x\in\mathbb{R}} \langle x\rangle^3 \big|\phi(x)\big| \\ &\leqslant \Big(\int_0^\infty \frac{1}{1+x^2}\,dx\Big) p_3(\phi) = \frac{\pi}{2}\,p_3(\phi)\,; \end{aligned}$$

thus

$$\big|\langle R, \phi\rangle\big| \leqslant \frac{\pi}{2}\,p_k(\phi) \qquad \forall\, k \geqslant 3\,.$$

Therefore, R is a tempered distribution.

2. For $\phi \in \mathscr{S}(\mathbb{R})$, define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} \dfrac{\phi(x) - \phi(0)}{x} & \text{if } x \neq 0\,, \\ \phi'(0) & \text{if } x = 0\,. \end{cases}$$

Then clearly ψ is continuous on $\mathbb{R}$, and

$$\sup_{x\in[-1,1]} \big|\psi(x)\big| \leqslant \sup_{x\in[-1,1]} \big|\phi'(x)\big| \leqslant p_1(\phi)\,.$$

By the fact that

$$\int_{-1}^1 \psi(x)\,dx = \lim_{\epsilon\to 0^+} \int_{[-1,1]\backslash(-\epsilon,\epsilon)} \psi(x)\,dx\,,$$

we find that

$$\begin{aligned} \langle T, \phi\rangle &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\,dx = \int_{\mathbb{R}\backslash[-1,1]} \frac{\phi(x)}{x}\,dx + \lim_{\epsilon\to 0^+} \int_{[-1,1]\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\,dx \\ &= \int_{\mathbb{R}\backslash[-1,1]} \frac{\phi(x)}{x}\,dx + \lim_{\epsilon\to 0^+} \int_{[-1,1]\backslash(-\epsilon,\epsilon)} \frac{\phi(x)-\phi(0)}{x}\,dx \\ &= \int_{\mathbb{R}\backslash[-1,1]} \frac{\phi(x)}{x}\,dx + \int_{-1}^1 \psi(x)\,dx\,. \end{aligned}$$

Therefore, $\langle T, \phi\rangle \in \mathbb{C}$ for all $\phi \in \mathscr{S}(\mathbb{R})$. Moreover,

$$\begin{aligned}\big|\langle T, \phi\rangle\big| &\leqslant \int_{\mathbb{R}\backslash[-1,1]} \Big|\frac{\phi(x)}{x}\Big|\,dx + \int_{-1}^{1} \big|\psi(x)\big|\,dx \leqslant \int_{\mathbb{R}\backslash[-1,1]} |x|^{-2}|x|\big|\phi(x)\big|\,dx + 2p_1(\phi)\\ &\leqslant \Big(2 + \int_{\mathbb{R}\backslash[-1,1]} |x|^{-2}\,dx\Big)p_1(\phi) = 4p_1(\phi)\,;\end{aligned}$$

thus $\big|\langle T, \phi\rangle\big| \leqslant 4p_k(\phi)$ for all $k \geqslant 1$. This implies that T is a tempered distribution.

3. Define $H_n(x) = \chi_{(0,n)}(x)$. For a Schwartz function $\phi \in \mathscr{S}(\mathbb{R})$, define $\psi = \phi - \phi(0)G$. Then $\psi \in \mathscr{S}(\mathbb{R})$, and

$$\begin{aligned}\langle \widehat{H}, \phi\rangle &= \langle \widehat{H}, \psi\rangle + \phi(0)\langle \widehat{H}, G\rangle = \langle H, \widehat{\psi}\rangle + \phi(0)\langle H, \widehat{G}\rangle\\ &= \lim_{n\to\infty} \langle H_n, \widehat{\psi}\rangle + \phi(0)\langle H, G\rangle\\ &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^n \Big(\int_{-\infty}^{\infty} \psi(x)e^{-ix\xi}dx\Big)d\xi + \sqrt{\frac{\pi}{2}}\phi(0)\end{aligned}$$

where we have used the fact that $\langle H, G\rangle = \sqrt{\dfrac{\pi}{2}}$ to conclude the last equality.

Define f by $f(x) = \dfrac{\psi(x)}{x}$ or to be more precise, $f(x) = \begin{cases} \dfrac{\psi(x)}{x} & \text{if } x \neq 0\,,\\ \psi'(0) & \text{if } x = 0\,,\end{cases}$. Then f is a Schwartz function. In fact, we have $\psi(x) = xf(x)$ for all $x \in \mathbb{R}$ and the Lebnitz rule implies that for $j \geqslant 0$,

$$xf^{(j)}(x) = \psi^{(j)}(x) - jf^{(j-1)}(x)$$

which implies that

$$|x|^k\big|f^{(j)}(x)\big| \leqslant |x|^k\big|\psi^{(j)}(x)\big| + k|x|^{k-1}\big|f^{(j-1)}(x)\big|$$

so that the boundedness of $|x|^k\big|f^{(j)}(x)\big|$ can be proved by induction.

By Fubini's Theorem,

$$\int_0^n \Big(\int_{-\infty}^{\infty} \psi(x)e^{-ix\xi}dx\Big)d\xi = \int_{-\infty}^{\infty} \Big(\int_0^n \psi(x)e^{-ix\xi}d\xi\Big)dx\,;$$

thus

$$\begin{aligned}\langle \widehat{H}, \phi\rangle &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\Big(\int_0^n e^{-ix\xi}d\xi\Big)dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \phi\rangle\\ &= \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\frac{1 - e^{-inx}}{ix}dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \phi\rangle\\ &= \frac{1}{\sqrt{2\pi}i} \int_{-\infty}^{\infty} \frac{\psi(x)}{x}\psi(x)\,dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \phi\rangle + i\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-inx}f(x)dx\\ &= \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\psi(x)}{x}\,dx + \sqrt{\frac{\pi}{2}}\,\langle \delta, \phi\rangle + i\lim_{n\to\infty} \widehat{f}(n)\,.\end{aligned}$$

Since $f \in \mathscr{S}(\mathbb{R})$, $\widehat{f} \in \mathscr{S}(\mathbb{R})$; thus $\lim\limits_{n\to\infty} \widehat{f}(n) = 0$. Therefore, by the fact G is an even function, we conclude that

$$\begin{aligned}
\langle \widehat{H}, \phi \rangle &= \lim_{\epsilon\to 0^+} \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{\psi(x)}{x}\, dx + \sqrt{\frac{\pi}{2}} \langle \delta, \phi \rangle \\
&= \lim_{\epsilon\to 0^+} \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\, dx + \sqrt{\frac{\pi}{2}} \langle \delta, \phi \rangle \\
&= \lim_{\epsilon\to 0} \frac{-i}{\sqrt{2\pi}} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{\phi(x)}{x}\, dx + \sqrt{\frac{\pi}{2}} \langle \delta, \phi \rangle = \langle T, \phi \rangle + \sqrt{\frac{\pi}{2}} \langle \delta, \phi \rangle ,
\end{aligned}$$

which shows that $\widehat{H} = \dfrac{-i}{\sqrt{2\pi}} T + \sqrt{\dfrac{\pi}{2}}\, \delta.$ □

Problem 12. The Hilbert transform of a function $f : \mathbb{R} \to \mathbb{R}$, denoted by $\mathscr{H}[f]$, is a function defined (formally) by

$$\mathscr{H}[f](x) = \frac{1}{\pi} \lim_{\epsilon\to 0^+} \int_{|y-x|>\epsilon} \frac{f(y)}{x-y}\, dy\,,$$

1. Show that $\mathscr{H}[f]$ is well-defined if $f \in \mathscr{S}(\mathbb{R})$.
2. Show that $\mathscr{F}\big[\mathscr{H}[f]\big](\xi) = i\mathrm{sgn}(\xi)\widehat{f}(\xi)$ for all $f \in \mathscr{S}(\mathbb{R})$.
3. Show that $\big\|\mathscr{H}[f]\big\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ for all $f \in \mathscr{S}(\mathbb{R})$, where $\|g\|_{L^2(\mathbb{R})} = \Big(\int_{\mathbb{R}} \big|g(x)\big|^2\, dx\Big)^{\frac{1}{2}}$.

Hint: Consider the tempered distribution T defined in Problem 11 by

$$\langle T, \phi \rangle = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\backslash[-\epsilon,\epsilon]} \frac{\phi(x)}{x}\, dx = \lim_{\epsilon\to 0^+} \Big(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \Big) \frac{\phi(x)}{x}\, dx \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,.$$

1. Show that $\mathscr{H}[f] = \langle T, \tau_x \widetilde{f} \rangle$ for all $f \in \mathscr{S}(\mathbb{R})$, where τ_x is a translation operator.
2. Show that the tempered distribution S defined by $\langle S, \phi \rangle = \langle T(x), x\phi(x) \rangle$ is indeed the same as the tempered distribution
$$\phi \mapsto \int_{\mathbb{R}} \phi(x)\, dx = \langle 1, \phi \rangle\,.$$
Use Problem 4 to show that $\dfrac{d}{d\xi}\widehat{T}(\xi) = -\sqrt{\dfrac{\pi}{2}} i \dfrac{d}{d\xi}\mathrm{sgn}(\xi)$, where sgn is given in Problem 6. Use the fact that $\dfrac{dT}{dx} = 0$ if and only if there exists C such that $\langle T, \phi \rangle = \langle C, \phi \rangle$ for all $\phi \in \mathscr{S}(\mathbb{R})$ to conclude that
$$\widehat{T}(\xi) = -\sqrt{\frac{\pi}{2}} i\mathrm{sgn}(\xi) + C$$
for some constant C. Find the constant C and also show that $\mathscr{H}[f] = \dfrac{1}{\pi} T * f = \sqrt{\dfrac{2}{\pi}} T \star f$.
3. Use the Plancherel formula.

Proof. 1. Let $f \in \mathscr{S}(\mathbb{R})$ be given. For each $\epsilon > 0$, the substitution of variable $z = x - y$ implies that

$$\int_{|y-x|>\epsilon} \frac{f(y)}{x-y}\,dy = \int_{|z|>\epsilon} \frac{f(x-z)}{z}\,dz = \int_{|z|>\epsilon} \frac{\widetilde{f}(z-x)}{z}\,dz = \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{(\tau_x \widetilde{f})(z)}{z}\,dz\,;$$

thus

$$\mathscr{H}[f](x) = \frac{1}{\pi}\lim_{\epsilon\to 0^+} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{(\tau_x \widetilde{f})(z)}{z}\,dz = \frac{1}{\pi}\langle T, \tau_x \widetilde{f}\rangle\,.$$

Since $\tau_x \widetilde{f}$ is also a Schwartz function for all $x \in \mathbb{R}$, by Problem 11 we conclude that $\mathscr{H}[f] : \mathbb{R} \to \mathbb{C}$ is a well-defined function.

2. Let S be defined by $\langle S, \phi\rangle = \langle T(x), x\phi(x)\rangle$ for all $\phi \in \mathscr{S}(\mathbb{R})$. Since $T \in \mathscr{S}(\mathbb{R})'$, there exists $\{C_k\}_{k=1}^{\infty}$ such that

$$\big|\langle T, \phi\rangle\big| \leqslant C_k p_k(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}) \text{ and } k \gg 1.$$

By the fact that

$$\begin{aligned} p_k\big(x\phi(x)\big) &= \sup_{x\in\mathbb{R}, 0\leqslant \ell\leqslant k} \langle x\rangle^k \Big|\frac{d^\ell}{dx^\ell}\big[x\phi(x)\big]\Big| = \sup_{x\in\mathbb{R}, 0\leqslant \ell\leqslant k} \langle x\rangle^k \Big|\ell\phi^{(\ell-1)}(x) + x\phi^{(\ell)}(x)\Big| \\ &\leqslant \sup_{x\in\mathbb{R}, 0\leqslant \ell\leqslant k} \langle x\rangle^k \Big|\ell\phi^{(\ell-1)}(x)\Big| + \sup_{x\in\mathbb{R}, 0\leqslant \ell\leqslant k} \langle x\rangle^{k+1} \big|\phi^{(\ell)}(x)\big| \leqslant (k+1)p_{k+1}(\phi)\,, \end{aligned}$$

we find that

$$\big|\langle S, \phi\rangle\big| \leqslant (k+1)C_k p_{k+1}(\phi) \qquad \forall\, \phi \in \mathscr{S}(\mathbb{R}) \text{ and } k \gg 1\,.$$

Therefore, S is a tempered distribution. Moreover, if $\phi \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned} \langle S, \phi\rangle = \langle T(x), x\phi(x)\rangle &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{x\phi(x)}{x}\,dx = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \phi(x)\,dx = \int_{\mathbb{R}} \phi(x)\,dx \\ &= \langle 1, \phi\rangle\,; \end{aligned}$$

thus we conclude that $S = 1$. Therefore, $\widehat{S} = \sqrt{2\pi}\,\delta$. By Problem 4,

$$\Big\langle \frac{d}{d\xi}\widehat{T}(\xi), \phi(\xi)\Big\rangle = -i\big\langle \mathscr{F}_x[xT(x)](\xi), \phi(\xi)\big\rangle = -i\langle \widehat{S}, \phi\rangle = -i\sqrt{2\pi}\langle \delta, \phi\rangle \quad \forall\, \phi \in \mathscr{S}(\mathbb{R})\,;$$

thus Problem 6 shows that

$$\frac{d}{d\xi}\widehat{T}(\xi) = -i\sqrt{2\pi}\frac{1}{2}\frac{d}{d\xi}\text{sgn}(\xi) = -\sqrt{\frac{\pi}{2}}\,i\frac{d}{d\xi}\text{sgn}(\xi)\,.$$

Therefore, there exists a constant C such that

$$\widehat{T}(\xi) = -\sqrt{\frac{\pi}{2}}\,i\,\text{sgn}(\xi) + C\,.$$

To determine C, let ϕ be a positive even Schwartz function (for example, $\phi(x) = e^{-x^2}$). Then $\widehat{\phi}$ is also an even Schwartz function so that

$$\langle \widehat{T}, \phi\rangle = \langle T, \widehat{\phi}\rangle = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\backslash(-\epsilon,\epsilon)} \frac{\widehat{\phi}(x)}{x}\,dx = 0$$

since the function $y = \dfrac{\widehat{\phi}(x)}{x}$ is odd. By the fact that $\langle \mathrm{sgn}, \phi \rangle = 0$, we find that

$$0 = \langle \widehat{T}, \phi \rangle = -\sqrt{\frac{\pi}{2}}\, i \,\langle \mathrm{sgn}, \phi \rangle + \langle C, \phi \rangle = C\langle 1, \phi \rangle\,.$$

Therefore, $C = 0$ which shows that $\widehat{T}(\xi) = -\sqrt{\frac{\pi}{2}} i\mathrm{sgn}(\xi)$.

Finally, if $f, \phi \in \mathscr{S}(\mathbb{R})$,

$$(\widetilde{f} * \phi)(x) = \int_{\mathbb{R}} \widetilde{f}(x-y)\phi(y)\,dy = \int_{\mathbb{R}} f(y-x)\phi(y)\,dy$$

so that

$$\langle T, \widetilde{f} * \phi \rangle = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} \int_{\mathbb{R}} \frac{\widetilde{f}(x-y)\phi(y)}{x}\,dy\,dx = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \Big(\int_{|x|>\epsilon} \frac{\widetilde{f}(x-y)}{x}\,dx \Big) \phi(y)\,dy\,.$$

Since

$$\begin{aligned}
\Big| \int_{|x|>\epsilon} \frac{f(y-x)}{x}\,dx \Big| &\leqslant \int_{\epsilon<|x|<1} \Big| \frac{f(y-x)}{x} \Big|\,dx + \lim_{R\to\infty} \int_{1<|x|<R} \Big| \frac{f(y-x)}{x} \Big|\,dx \\
&\leqslant \int_{\epsilon<|x|<1} \Big| \frac{f(y-x)-f(y)}{x} \Big|\,dx + \lim_{R\to\infty} \int_{1<|x|<R} \Big| \frac{(x-y)f(y-x)}{x^2} \Big|\,dx \\
&\quad + \lim_{R\to\infty} \int_{1<|x|<R} \Big| \frac{yf(y-x)}{x^2} \Big|\,dx \\
&\leqslant 2\sup_{x\in\mathbb{R}} |f'(x)| + 2\Big(\sup_{x\in\mathbb{R}} |xf(x)| + |y| \sup_{x\in\mathbb{R}} |f(x)| \Big) \int_1^\infty x^{-2}\,dx \\
&\leqslant 4p_1(f) + 2|y|p_0(f)\,,
\end{aligned}$$

the Dominated Convergence Theorem implies that

$$\begin{aligned}
\langle T, \widetilde{f} * \phi \rangle &= \int_{\mathbb{R}} \lim_{\epsilon\to 0^+} \Big(\int_{|x|>\epsilon} \frac{\widetilde{f}(x-y)}{x}\,dx \Big) \phi(y)\,dy = \int_{\mathbb{R}} \lim_{\epsilon\to 0^+} \Big(\int_{|y-x|>\epsilon} \frac{f(z)}{y-z}\,dz \Big) \phi(y)\,dy \\
&= \int_{\mathbb{R}} \pi \mathscr{H}[f](y)\phi(y)\,dy = \big\langle \pi\mathscr{H}[f], \phi \big\rangle\,.
\end{aligned}$$

Therefore, by the definition of the convolution, we conclude that $\mathscr{H}[f] = \dfrac{1}{\pi}(T * f) = \sqrt{\dfrac{2}{\pi}}(T \star f)$ and this further implies that

$$\mathscr{F}\big[\mathscr{H}[f]\big](\xi) = \sqrt{\frac{2}{\pi}}\,\widehat{T}(\xi)\widehat{f}(\xi) = -i\mathrm{sgn}(\xi)\widehat{f}(\xi)\,.$$

3. By the Plancherel formula,

$$\big\|\mathscr{H}[f]\big\|_{L^2(\mathbb{R})}^2 = \big\|\mathscr{F}\big[\mathscr{H}[f]\big]\big\|_{L^2(\mathbb{R})}^2 = \|i\mathrm{sgn} f\|_{L^2(\mathbb{R})}^2 = \|f\|_{L^2(\mathbb{R})}^2$$

which shows that $\big\|\mathscr{H}[f]\big\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ for all $f \in \mathscr{S}(\mathbb{R})$. □