Exercise Problem Sets 10

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Problem 1. This problem contributes to another proof of showing that the Fourier series of f converges uniformly to f on $\mathbb{R}$ if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for $\frac{1}{2} < \alpha \leqslant 1$. Complete the following.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic such that f is Riemann integrable on $[-\pi, \pi]$. Show that

$$\widehat{f}_k = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(x + \tfrac{\pi}{k}\big) e^{-ikx}\, dx$$

   and hence

$$\widehat{f}_k = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big[f(x) - f\big(x + \tfrac{\pi}{k}\big)\big] e^{-ikx}\, dx\,.$$

   Therefore, if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies $|\widehat{f}_k| \leqslant \dfrac{\pi^\alpha \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic such that f is Riemann integrable on $[-\pi, \pi]$. Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \big|f(x+h) - f(x-h)\big|^2\, dx = \sum_{k=-\infty}^{\infty} 4\sin^2(kh) |\widehat{f}_k|^2\,.$$

   Therefore, if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies

$$\sum_{k=-\infty}^{\infty} \sin^2(kh) |\widehat{f}_k|^2 \leqslant \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} 2^{2(\alpha-1)} |h|^{2\alpha} \tag{0.1}$$

3. Let $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, and $p \in \mathbb{N}$. Show that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k|^2 \leqslant \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{2\alpha}}{2^{2\alpha p + 1}}\,.$$

   Hint: Let $h = \dfrac{\pi}{2^{p+1}}$ in (0.1).

4. Show that if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\frac{1}{2} < \alpha \leqslant 1$, then $\sum\limits_{k=-\infty}^{\infty} |\widehat{f}_k| < \infty$; thus Problem 8 of Exercise 7 implies that the Fourier series of f converges uniformly to f on $\mathbb{R}$.

Proof. 1. By substitution of variables,

$$\widehat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky}\, dy \overset{\text{“}y = x + \frac{\pi}{k}\text{”}}{=} \frac{1}{2\pi} \int_{-\pi-\frac{\pi}{k}}^{\pi - \frac{\pi}{k}} f\big(x + \frac{\pi}{k}\big) e^{-ikx} e^{-i\pi}\, dx$$

so that the periodicity of f and the function $y = e^{-ikx}$ implies that

$$\widehat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(x + \frac{\pi}{k}\big) e^{-ikx} e^{-i\pi}\, dx = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(x + \frac{\pi}{k}\big) e^{-ikx}\, dx\,.$$

Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0,1]$. Then

$$\big|f(x) - f(y)\big| \leqslant \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} |x-y|^\alpha \qquad \forall\, x, y \in \mathbb{R}\,.$$

Therefore,

$$\big|f(x + \frac{\pi}{k} - f(x)\big| \leqslant \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \frac{\pi^\alpha}{k^\alpha}$$

and we then conclude that

$$\big|\widehat{f}_k\big| \leqslant \frac{1}{4\pi}\int_{-\pi}^{\pi} \big|f(x) - f\big(x + \frac{\pi}{k}\big)\big|\,dx \leqslant \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^\alpha}{4\pi k^\alpha}\int_{-\pi}^{\pi} dx = \frac{\pi^\alpha \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}\,.$$

2. For $h \neq 0$, let $g(x) = f(x+h) - f(x-h)$. Then by substitution of variables,

$$\begin{aligned}\widehat{g}_k &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(y+h)e^{-iky}\,dy - \int_{-\pi}^{\pi} f(y-h)e^{-iky}\,dy\Big]\\ &= \frac{1}{2\pi}\Big[\int_{-\pi+h}^{\pi+h} f(x)e^{-ikx}e^{ikh}\,dx - \int_{-\pi-h}^{\pi-h} f(x)e^{-ikx}e^{-ikh}\,dx\Big]\end{aligned}$$

so that the periodicity of f and the function $y = e^{-ikx}$ implies that

$$\begin{aligned}\widehat{g}_k &= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(x)e^{-ikx}e^{ikh}\,dx - \int_{-\pi-h}^{\pi-h} f(x)e^{-ikx}e^{-ikh}\,dx\Big]\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\big(e^{ikh} - e^{-ikh}\big)\,dx = \frac{2i\sin(kh)}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx = 2i\sin(kh)\widehat{f}_k\,.\end{aligned}$$

Therefore, the Parseval identity shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x+h) - f(x-h)\big|^2\,dx = \sum_{k=-\infty}^{\infty}\big|\widehat{g}_k\big|^2 = \sum_{k=-\infty}^{\infty} 4\sin^2(kh)\big|\widehat{f}_k\big|^2\,.$$

If in addition $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, then the identity above implies that

$$\sum_{k=-\infty}^{\infty} 4\sin^2(kh)\big|\widehat{f}_k\big|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}h^{2\alpha}\,dx = \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}(2h)^{2\alpha}$$

which verifies (0.1).

3. For each $p \in \mathbb{N}$, letting $h = \dfrac{\pi}{2^{p+1}}$ in (0.1) we find that

$$\sum_{2^{p-1}\leqslant|k|<2^p} \sin^2\frac{k\pi}{2^{p+1}}\big|\widehat{f}_k\big|^2 \leqslant \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}2^{2(\alpha-1)}\frac{\pi^{2\alpha}}{2^{2(p+1)\alpha}} = \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^{2\alpha}}{2^{2(p\alpha+1)}}$$

Since for $2^{p-1} \leqslant |k| < 2^p$, $\sin^2\dfrac{k\pi}{2^{p+1}} \geqslant \dfrac{1}{2}$, the inequality above implies that

$$\sum_{2^{p-1}\leqslant|k|<2^p}\big|\widehat{f}_k\big|^2 \leqslant \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^{2\alpha}}{2^{2p\alpha+1}}\,.$$

4. Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0.5, 1]$. For each $p \in \mathbb{N}$, by the Cauchy inequality and the result in part 3 we obtain that

$$\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k| \leqslant \Big(\sum_{2^{p-1} \leqslant |k| < 2^p} 1 \Big)^{\frac{1}{2}} \Big(\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k|^2 \Big)^{\frac{1}{2}} = \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^\alpha}{2^{p(\alpha - \frac{1}{2}) + 1}}.$$

Therefore, by the fact that $\sum_{p=1}^{\infty} \frac{1}{2^{p(\alpha - \frac{1}{2})}} < \infty$ (since $\alpha > \frac{1}{2}$), we conclude that

$$\sum_{k=-\infty}^{\infty} |\widehat{f}_k| = |\widehat{f}_0| + \sum_{p=1}^{\infty} \sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k| \leqslant |\widehat{f}_0| + \frac{\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^\alpha}{2} \sum_{p=1}^{\infty} \frac{1}{2^{p(\alpha - \frac{1}{2})}} < \infty ;$$

thus Problem 8 of Exercise 7 implies that the Fourier series of f converges uniformly to f on $\mathbb{R}$ if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\alpha \in (0.5, 1]$. □

Problem 2. Let $f : [0, L] \to \mathbb{R}$ be a square integrable function.

1. Suppose that $\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos \frac{k\pi x}{L}$ is the cosine series of f. Find $\sum_{k=1}^{\infty} c_k^2$ in terms of integrals of f and f^2.

2. Suppose that $\sum_{k=1}^{\infty} s_k \cos \frac{k\pi x}{L}$ is the sine series of f. Find $\sum_{k=1}^{\infty} s_k^2$ in terms of integral of f^2.

Proof. 1. Let $\bar{f}$ be the even extension of f; that is, $\bar{f} : (-L, L) \to \mathbb{R}$ is given by $\bar{f}(x) = f(x)$ if $x \in [0, L)$ and $\bar{f}(x) = f(-x)$ if $x \in (-L, 0)$. Then the cosine series of f is the Fourier series of $\bar{f}$ so that

$$\text{the cosine series of } f = s(\bar{f}, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos \frac{k\pi x}{L},$$

where

$$c_k = \frac{1}{L} \int_{-L}^{L} \bar{f}(x) \cos \frac{k\pi x}{L} \, dx = \frac{2}{L} \int_0^L f(x) \cos \frac{k\pi x}{L} \, dx .$$

Define $g : (-\pi, \pi) \to \mathbb{R}$ by $g(x) = f\big(\frac{Lx}{\pi}\big)$. Then the Fourier coefficients of g is the same as the Fourier coefficients of $\bar{f}$. By the Parseval identity,

$$\frac{c_0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} c_k^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)^2 \, dx = \frac{1}{2L} \int_{-L}^{L} \bar{f}(x)^2 \, dx = \frac{1}{L} \int_0^L f(x)^2 \, dx ;$$

thus

$$\sum_{k=1}^{\infty} c_k^2 = 2\Big[\frac{1}{L} \int_0^L f(x)^2 \, dx - \frac{c_0^2}{4}\Big] = \frac{2}{L} \int_0^L f(x)^2 \, dx - \frac{1}{2}\Big(\frac{2}{L} \int_0^L f(x) \, dx\Big)^2 .$$

2. Let $\bar{f}$ be the odd extension of f; that is, $\bar{f} : (-L, L) \to \mathbb{R}$ is given by $\bar{f}(x) = f(x)$ if $x \in [0, L)$ and $\bar{f}(x) = -f(-x)$ if $x \in (-L, 0)$. Then the sine series of f is the Fourier series of $\bar{f}$ so that

$$\text{the sine series of } f = s(\bar{f}, x) = \sum_{k=1}^{\infty} s_k \cos \frac{k\pi x}{L},$$

where

$$s_k = \frac{1}{L}\int_{-L}^{L} \bar{f}(x)\sin\frac{k\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx\,.$$

Define $g:(-\pi,\pi)\to\mathbb{R}$ by $g(x) = f\big(\frac{Lx}{\pi}\big)$. Then the Fourier coefficients of g is the same as the Fourier coefficients of $\bar{f}$. By the Parseval identity,

$$\frac{1}{2}\sum_{k=1}^{\infty} s_k^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)^2\,dx = \frac{1}{2L}\int_{-L}^{L}\bar{f}(x)^2\,dx = \frac{1}{L}\int_0^L f(x)^2\,dx\,;$$

thus

$$\sum_{k=1}^{\infty} s_k^2 = \frac{2}{L}\int_0^L f(x)^2\,dx\,.$$

□

Problem 3. Find the Fourier transform of the following functions.

1. $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = xe^{-tx^2}$ for $t>0$.
2. $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = \chi_{(-a,a)}(x)$, the characteristic (indicator) function of the set $(-a,a)$.
3. $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = \begin{cases} e^{-tx} & \text{if } x>0\,,\\ 0 & \text{if } x\leqslant 0\,,\end{cases}$ where $t>0$.

Solution. 1. Integrating by parts,

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} xe^{-tx^2}e^{-ix\xi}\,dx = \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\int_{-R}^{R} xe^{-tx^2}e^{-ix\xi}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\Big[\frac{-1}{2t}e^{-tx^2}e^{-ix\xi}\Big|_{x=-R}^{x=R} - \frac{i\xi}{2t}\int_{-R}^{R} e^{-tx^2}e^{-ix\xi}\,dx\Big]\\ &= -\frac{i\xi}{2t}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-tx^2}e^{-ix\xi}\,dx = -\frac{i\xi}{2t}\mathscr{F}_x\big[e^{-tx^2}\big](\xi)\,.\end{aligned}$$

Noting that with $P_t(x) = \frac{1}{\sqrt{t}}e^{-\frac{x^2}{2t}}$, the formula (**??**) implies that

$$\mathscr{F}_x\big[e^{-tx^2}\big](\xi) = \frac{1}{\sqrt{2t}}\widehat{P_{\frac{1}{2t}}}(\xi) = \frac{1}{\sqrt{2t}}e^{-\frac{\xi^2}{4t}}\,;$$

thus $\widehat{f}(\xi) = -\frac{i\xi}{\sqrt{2t}^3}e^{-\frac{\xi^2}{4t}}$.

2. We integrate directly and obtain that if $\xi\neq 0$

$$\begin{aligned}\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ix\xi}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\big[\cos(x\xi) - i\sin(x\xi)\big]\,dx\\ &= \frac{1}{\sqrt{2\pi}\xi}\big[\sin(x\xi) + i\cos(x\xi)\big]\Big|_{x=-a}^{x=a} = \frac{2\sin(a\xi)}{\sqrt{2\pi}\xi}\,,\end{aligned}$$

while $\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} 1\,dx = \frac{2a}{\sqrt{2\pi}}$. Therefore,

$$\widehat{f}(\xi) = \begin{cases}\dfrac{2\sin(a\xi)}{\sqrt{2\pi}\xi} & \text{if } \xi\neq 0\,,\\[2ex] \dfrac{2a}{\sqrt{2\pi}} & \text{if } \xi = 0\,.\end{cases}$$

3. Since $t > 0$, $\lim_{R\to\infty} e^{-(t+i\xi)R} = 0$ for all $\xi \in \mathbb{R}$; thus

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-tx} e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_0^R e^{-(t+i\xi)x}\, dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \frac{e^{-(t+i\xi)x}}{-(t+i\xi)}\Big|_{x=0}^{x=R}$$
$$= \frac{1}{\sqrt{2\pi}(t+i\xi)}$$

□

Problem 4. Let $\alpha > 0$ be given. Show that the Fourier transform of the function

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} dt$$

is positive.

Proof. For $\xi \in \mathbb{R}^n$, define $g(x,t) = t^{\alpha-1} e^{-t} e^{-t|x|^2} e^{ix\cdot\xi}$. By the Tonelli Theorem,

$$\int_{\mathbb{R}^n \times (0,\infty)} \big|g(x,t)\big|\, d(x,t) = \int_{\mathbb{R}} \int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} dt dx = \int_0^\infty t^{\alpha-1} e^{-t} \Big(\int_{\mathbb{R}} e^{-t|x|^2}\, dx \Big) dt$$
$$= \int_0^\infty t^{\alpha-1} e^{-t} \Big(\frac{t}{\pi}\Big)^{\frac{n}{2}} dt = \frac{1}{\sqrt{\pi}^n} \int_0^\infty t^{\frac{n}{2}+\alpha-1} e^{-t}\, dt = \frac{\Gamma(\frac{n}{2}+\alpha-1)}{\sqrt{\pi}^n} < \infty\,.$$

The computation above also shows that $f \in L^1(\mathbb{R}^n)$. Therefore, the Fubini Theorem implies that

$$\Gamma(a)\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \Big(\int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} dt \Big) e^{-ix\cdot\xi} dx$$
$$= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \Big(\int_0^\infty t^{\alpha-1} e^{-t} e^{-t|x|^2} e^{-ix\cdot\xi} dt \Big) dx = \int_0^\infty t^{\alpha-1} e^{-t} \Big(\frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-t|x|^2} e^{-ix\cdot\xi} dx \Big) dt$$
$$= \int_0^\infty t^{\alpha-1} e^{-t} \mathscr{F}_x[e^{-t|x|^2}](\xi)\, dt = \int_0^\infty t^{\alpha-1} e^{-t} \sqrt{2t}^n e^{-\frac{|\xi|^2}{4t}}\, dt > 0\,.$$

The positivity of $\widehat{f}$ then follows from the fact that $\Gamma(\alpha) > 0$. □