# Exercise Problem Sets 7

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**Problem 1.** 1. Let $f : [-\pi, \pi]$ be a Riemann integrable function. Show that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\cos kx\,dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0\,.$$

2. Show the Riemann-Lebesgue Lemma

> If $f : [-\pi, \pi] \to \mathbb{R}$ is an integrable function, then
> $$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\cos kx\,dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0\,.$$

**Hint**: First show that for every $\varepsilon > 0$ there exists a Riemann integrable function $g : [-\pi, \pi] \to \mathbb{R}$ such that $\displaystyle\int_{-\pi}^{\pi} \big|f(x) - g(x)\big|\,dx < \varepsilon$, then apply the conclusion in 1.

*Proof.* 1. Let $\varepsilon > 0$ be given. Then by Lemma 6.63 in the lecture note, there exists $g \in \mathscr{C}([-\pi, \pi]; \mathbb{R})$ such that

$$f(x) \leqslant g(x) \leqslant \sup_{x\in[-\pi,\pi]} f(x) \quad \forall\, x \in [-\pi, \pi] \qquad \text{and} \qquad \int_{-\pi}^{\pi} f(x)\,dx > \int_{-\pi}^{\pi} g(x)\,dx - \frac{\varepsilon}{3}\,.$$

By the Weierstrass Theorem, there exists a polynomial $p$ such that

$$\|g - p\|_\infty < \frac{\varepsilon}{6\pi}\,.$$

Since $p$ is a polynomial, integrating by parts (or by Exercise Problem **??**) we can show that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} p(x)\cos kx\,dx = \lim_{k\to\infty} \int_{-\pi}^{\pi} p(x)\sin kx\,dx = 0\,.$$

Therefore, there exists $N > 0$ such that if $k \geqslant N$,

$$\Big|\int_{-\pi}^{\pi} p(x)\cos kx\,dx\Big| < \frac{\varepsilon}{3} \qquad \text{and} \qquad \Big|\int_{-\pi}^{\pi} p(x)\sin kx\,dx\Big| < \frac{\varepsilon}{3}\,.$$

Therefore, if $k \geqslant N$,

$$\begin{aligned}
\Big|\int_{-\pi}^{\pi} f(x)\cos kx\,dx\Big| &\leqslant \Big|\int_{-\pi}^{\pi} \big[f(x) - g(x)\big]\cos kx\,dx\Big| + \Big|\int_{-\pi}^{\pi} \big[g(x) - p(x)\big]\cos kx\,dx\Big| \\
&\quad + \Big|\int_{-\pi}^{\pi} p(x)\cos kx\,dx\Big| \\
&\leqslant \int_{-\pi}^{\pi} \big|f(x) - g(x)\big|\,dx + \int_{-\pi}^{\pi} \|g - p\|_\infty\,dx + \frac{\varepsilon}{3} \\
&\leqslant \int_{-\pi}^{\pi} \big[g(x) - f(x)\big]\,dx + \int_{-\pi}^{\pi} \frac{\varepsilon}{6\pi}\,dx + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,,
\end{aligned}$$

and similarly,

$$\Big|\int_{-\pi}^{\pi} f(x)\cos kx\,dx\Big| < \varepsilon \quad \text{whenever} \quad k \geqslant N\,.$$

2. Let $g_k(x) = (f^+ \wedge k)(x) - (f^- \wedge k)(x)$. Then

$$\begin{aligned}\int_{-\pi}^{\pi} \big|f(x) - g_k(x)\big|\,dx &= \int_{-\pi}^{\pi} \big|f^+(x) - f^-(x) - g_k(x)\big|\,dx \\ &\leqslant \int_{-\pi}^{\pi} \big|f^+(x) - (f^+ \wedge k)(x)\big|\,dx + \int_{-\pi}^{\pi} \big|f^-(x) - (f^- \wedge k)(x)\big|\,dx\,;\end{aligned}$$

thus by the fact that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} (f^+ \wedge k)(x)\,dx = \int_{-\pi}^{\pi} f^+(x)\,dx \quad\text{and}\quad \lim_{k\to\infty} \int_{-\pi}^{\pi} (f^- \wedge k)(x)\,dx = \int_{-\pi}^{\pi} f^-(x)\,dx\,,$$

we find that there exists $K > 0$ such that

$$\int_{-\pi}^{\pi} \big|f(x) - g_k(x)\big|\,dx < \frac{\varepsilon}{2} \qquad\text{whenever}\qquad k \geqslant K\,.$$

Let $h = g_K$. Note that $h$ is Riemann integrable on $[-\pi, \pi]$; thus part 1 implies that there exists $N > 0$ such that if $k \geqslant N$,

$$\Big|\int_{-\pi}^{\pi} h(x)\cos kx\,dx\Big| < \frac{\varepsilon}{2} \qquad\text{and}\qquad \Big|\int_{-\pi}^{\pi} h(x)\sin kx\,dx\Big| < \frac{\varepsilon}{2}\,.$$

Therefore, if $k \geqslant N$,

$$\begin{aligned}\Big|\int_{-\pi}^{\pi} f(x)\cos kx\,dx\Big| &= \Big|\int_{-\pi}^{\pi} \big[f(x) - h(x)\big]\cos kx\,dx\Big| + \Big|\int_{-\pi}^{\pi} h(x)\cos kx\,dx\Big| \\ &\leqslant \int_{-\pi}^{\pi} \big|f(x) - h(x)\big|\,dx + \Big|\int_{-\pi}^{\pi} h(x)\cos kx\,dx\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and similarly,

$$\Big|\int_{-\pi}^{\pi} f(x)\sin kx\,dx\Big| < \varepsilon \qquad\text{whenever}\qquad k \geqslant N\,.$$

□

**Problem 2.** Suppose that $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$; that is, $f$ is $2\pi$-periodic Hölder continuous function with exponent $\alpha$ for some $\alpha \in (0, 1]$. Show that (without using the Berstein Theorem) the Fourier series of $f$ converges pointwise to $f$, by completing the following.

1. Explain why it is enough to show that $s_n(f, 0) \to f(0)$ as $n \to \infty$. Also explain why we can assume that $f(0) = 0$.

2. Show that
$$\lim_{n\to\infty} \Big(s_n(f, 0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx\Big) = 0\,.$$
Therefore, it suffices to show that $\lim\limits_{n\to\infty} \displaystyle\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx = 0$ if $f(0) = 0$.

3. Show that if $f \in \mathscr{C}^{0,\alpha}(\mathbb{R})$ and $f(0) = 0$, then the function $y = \dfrac{f(x)}{x}$ is integrable. Apply the Riemann-Lebesgue Lemma to conclude that $s_n(f, 0) \to 0$ as $n \to \infty$.

*Proof.* 1. Suppose that one can show that if $g$ is a $2\pi$-periodic Hölder continuous function with exponent $\alpha \in (0,1]$, then $s_n(g,0) \to g(0)$ as $n \to \infty$. If $f$ is $2\pi$-periodic Hölder continuous function with exponent $\alpha \in (0,1]$ and $a \in \mathbb{R}$, let $g(x) = f(x+a)$. Then $g$ is a $2\pi$-periodic Hölder continuous function with exponent $\alpha$; thus $s_n(g,0) \to g(0)$ as $n \to \infty$.

On the other hand, let $\{c_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ be the Fourier coefficients of $f$ and $\{\bar{c}_k\}_{k=0}^{\infty}$ and $\{\bar{s}_k\}_{k=1}^{\infty}$ be the Fourier coefficients of $g$. Then

$$\begin{aligned}\bar{c}_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+a)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos k(x-a)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\big(\cos kx\cos ka + \sin kx \sin ka\big)\,dx \\ &= c_k\cos ka + s_k \sin ka\,.\end{aligned}$$

Note that

$$s_n(g,0) = \frac{\bar{c}_0}{2} + \sum_{k=1}^{n}\big[\bar{c}_k\cos(k\cdot 0) + \bar{s}_k\sin(k\cdot 0)\big] = \sum_{k=1}^{n}\big(c_k\cos ka + s_k\sin ka\big) = s_n(f,a)\,;$$

thus the fact that $g(0) = f(a)$ implies that $s_n(f,a) \to f(a)$ as $n \to \infty$. Moreover, if $f(0) \neq 0$, we consider the function $h(x) = f(x) - f(0)$. Then $h(0) = 0$ and $s_n(f,x) = s_n(h,x) + f(0)$ so that if the $s_n(h,0)$ converges to 0, then $s_n(f,0)$ converges to $f(0)$. In other words, we can further assume that $f(0) = 0$.

2. Note that $s_n(f,x) = (D_n \star f)(x)$; thus

$$s_n(f,0) = \int_{-\pi}^{\pi} f(x)\frac{\sin(n+\frac{1}{2})x}{2\pi\sin\frac{x}{2}}\,dx\,.$$

Therefore,

$$\begin{aligned}s_n(f,0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big[\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}} - \frac{\sin nx}{x}\Big]\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big(\frac{\sin nx\cos\frac{x}{2} + \sin\frac{x}{2}\cos nx}{2\sin\frac{x}{2}} - \frac{\sin nx}{x}\Big)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)\sin nx\,dx\,.\end{aligned}$$

Note that

$$\lim_{x\to 0}\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big) = \lim_{x\to 0}\frac{x\cos\frac{x}{2} - 2\sin\frac{x}{2}}{2x\sin\frac{x}{2}} = \lim_{x\to 0}\frac{x(1-\frac{x^2}{8}) - 2(\frac{x}{2} - \frac{x^3}{48})}{2x\cdot\frac{x}{2}} = 0\,;$$

thus the function $y = f(x)\Big(\dfrac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \dfrac{1}{x}\Big)$ is continuous on $[-\pi,\pi]$. By the Riemann-Lebesgue Lemma,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\Big(\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} - \frac{1}{x}\Big)\sin nx\,dx = 0\,.$$

Therefore,

$$\lim_{n\to\infty}\Big(s_n(f,0) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin nx}{x}\,dx\Big) = 0\,.$$

□

**Problem 3** (此題太早放，有些背景知識不足，等 §8.6 上完之後再出一次習題並再給解答)**.** This problem contributes to another proof of showing that the Fourier series of $f$ converges uniformly to $f$ on $\mathbb{R}$ if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for $\frac{1}{2} < \alpha \leqslant 1$. Complete the following.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic such that $f$ is Riemann integrable on $[-\pi, \pi]$. Show that
$$\widehat{f}_k = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\big(x + \frac{\pi}{k}\big) e^{-ikx}\, dx$$
and hence
$$\widehat{f}_k = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big[f(x) - f\big(x + \frac{\pi}{k}\big)\big] e^{-ikx}\, dx\,.$$
Therefore, if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies $|\widehat{f}_k| \leqslant \dfrac{\pi^\alpha \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be $2\pi$-periodic such that $f$ is Riemann integrable on $[-\pi, \pi]$. Show that
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \big|f(x+h) - f(x-h)\big|^2 dx = \sum_{k=-\infty}^{\infty} 4 \sin^2(kh) |\widehat{f}_k|^2\,.$$
Therefore, if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies
$$\sum_{k=-\infty}^{\infty} \sin^2(kh)|\widehat{f}_k|^2 \leqslant \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} 2^{2(\alpha-1)} |h|^{2\alpha} \tag{0.1}$$

3. Let $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, and $p \in \mathbb{N}$. Show that
$$\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k|^2 \leqslant \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} \pi^{2\alpha}}{2^{2\alpha p + 1}}\,.$$
**Hint**: Let $h = \dfrac{\pi}{2^{p+1}}$ in (0.1).

4. Show that if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\frac{1}{2} < \alpha \leqslant 1$, then $\sum\limits_{k=-\infty}^{\infty} |\widehat{f}_k| < \infty$; thus Problem 8 implies that the Fourier series of $f$ converges uniformly to $f$ on $\mathbb{R}$.

**Problem 4.** Prove Lemma 8.15 in the lecture note.

*Proof.* It suffices to show that
$$\sup_{|x-y| \leqslant \delta_1} \big|f(x) - f(y)\big| \leqslant \Big(1 + \frac{\delta_1}{\delta_2}\Big) \sup_{|x-y| \leqslant \delta_2} \big|f(x) - f(y)\big|$$
for $\delta_1 > \delta_2$. Suppose that $\delta_1 > \delta_2 > 0$, and $|x - y| < \delta_1$. Let $N$ be the largest integer smaller than $\dfrac{\delta_1}{\delta_2}$; that is, $N$ is the unique integer satisfying that
$$\frac{\delta_1}{\delta_2} - 1 \leqslant N < \frac{\delta_1}{\delta_2}\,. \tag{$\star$}$$

Then there exist $N$ points $x_1, x_2, \cdots, x_N$ such that $x < x_1 < x_2 < \cdots < x_N < y$ and

$$|x - x_1| < \delta_2\,, \quad |y - x_N| < \delta_2 \qquad \text{and} \qquad |x_i - x_{i+1}| < \delta_2 \quad \forall\, 1 \leqslant i \leqslant N-1\,;$$

thus the triangle inequality implies that

$$\begin{aligned}\big|f(x) - f(y)\big| &\leqslant \big|f(x) - f(x_1)\big| + \big|f(y) - f(x_N)\big| + \sum_{i=1}^{N-1} \big|f(x_i) - f(x_{i+1})\big| \\ &\leqslant (N+1) \sup_{|x-y|\leqslant\delta_2} \big|f(x) - f(y)\big|\,.\end{aligned}$$

The desired inequality then follows from $(\star)$. □

**Problem 5.** Let $f$ be a $2\pi$-periodic Lipchitz function. Show that for $n \geqslant 2$,

$$\|f - F_{n-1} \star f\|_\infty \leqslant \frac{1 + 2\log n}{2n} \pi \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} \tag{0.2}$$

and

$$\big\|f - s_n(f, \cdot)\big\|_\infty \leqslant \frac{2\pi(1 + \log n)^2}{n} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}\,. \tag{0.3}$$

**Hint**: For (0.2), apply the estimate

$$F_n(x) \leqslant \min\Big\{\frac{n+1}{2\pi}, \frac{\pi}{2(n+1)x^2}\Big\}$$

in the following inequality:

$$\big|f(x) - F_{n-1} \star f(x)\big| \leqslant \Big[\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big] \big|f(x) - f(x-y)\big| F_{n-1}(y)\, dy$$

with $\delta = \dfrac{\pi}{n}$. For (0.3), use (8.2.7) in the lecture note and note that

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant \|f - F_n \star f\|_\infty\,.$$

*Proof.* Recall that the Fejér kernel $F_n$ is given by

$$F_n(x) = \begin{cases} \dfrac{1}{2\pi(n+1)} \dfrac{\sin^2 \frac{(n+1)x}{2}}{\sin^2 \frac{x}{2}} & \text{if } x \notin \big\{2k\pi \,\big|\, k \in \mathbb{Z}\big\}\,, \\ \dfrac{n+1}{2\pi} & \text{if } x \in \big\{2k\pi \,\big|\, k \in \mathbb{Z}\big\}\,. \end{cases}$$

Therefore, by the fact that $\sin|x| \geqslant \dfrac{2}{\pi}|x|$ for $|x| < \dfrac{\pi}{2}$, we find that

$$F_n(x) \leqslant \min\Big\{\frac{n+1}{2\pi}, \frac{\pi}{2(n+1)x^2}\Big\}\,.$$

By the fact that $\displaystyle\int_{-\pi}^{\pi} F_{n-1}(x)\, dx = 0$ for all $n \geqslant 2$, we find that if $n \geqslant 2$ and $0 < \delta < \pi$,

$$\begin{aligned}\big|f(x) - F_{n-1} \star f(x)\big| &= \Big|\int_{-\pi}^{\pi} f(x) F_{n-1}(x-y)\, dy - \int_{-\pi}^{\pi} f(y) F_{n-1}(x-y)\, dy\Big| \\ &= \Big|\int_{-\pi}^{\pi} \big[f(x) - f(y)\big] F_{n-1}(x-y)\, dy\Big| \\ &= \Big|\int_{-\pi}^{\pi} \big[f(x) - f(x-y)\big] F_{n-1}(y)\, dy\Big| \\ &= \Big|\Big(\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big) \big[f(x) - f(x-y)\big] F_{n-1}(y)\, dy\Big|\,.\end{aligned}$$

Let $\delta = \dfrac{\pi}{n}$. Then

$$\left|\int_{-\delta}^{\delta} \big[f(x) - f(x-y)\big] F_{n-1}(y)\, dy\right| \leqslant \int_{-\delta}^{\delta} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} |y| \cdot \frac{n}{2\pi}\, dy = \frac{n\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{\pi} \int_0^{\delta} y\, dy$$
$$= \frac{n\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{2\pi} \frac{\pi^2}{n^2} = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{2n}\,.$$

Moreover,

$$\left|\int_{\delta \leqslant |y| \leqslant \pi} \big[f(x) - f(x-y)\big] F_{n-1}(y)\, dy\right| \leqslant \int_{\delta \leqslant |y| \leqslant \pi} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} |y| \cdot \frac{\pi}{2ny^2}\, dy$$
$$= \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{n} \int_{\delta}^{\pi} \frac{1}{y}\, dy = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}}{n} \log\frac{\pi}{\delta} = \frac{\pi\|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} \log n}{n}\,.$$

The two inequalities above implies (0.2).

For the validity of (0.3), by the fact that

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant \|f - F_n \star f\|_\infty$$

we conclude from (0.2) and (8.2.7) in the lecture note that

$$\big\|f - s_n(f,\cdot)\big\|_\infty \leqslant (3 + \log n)\|f - F_n \star f\|_\infty \leqslant \frac{(3+\log n)(1 + 2\log(n+1))}{2(n+1)} \pi \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})}$$

and the desired inequality follows from the fact that

$$\frac{(3+\log n)(1 + 2\log(n+1))}{2(n+1)} \leqslant \frac{2(1+\log n)^2}{n} \qquad \forall\, n \geqslant 2\,. \qquad \square$$

**Problem 6.** In this problem, we are concerned with the following

**Theorem 0.1** (Bernstein)**.** *Suppose that $f$ is a $2\pi$-periodic function such that for some constant $C$ and $\alpha \in (0,1)$,*

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_\infty \leqslant C n^{-\alpha}$$

*for all $n \in \mathbb{N}$. Then $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$.*

Complete the following to prove the theorem.

1. Show that
$$\|p'\|_\infty \leqslant n\|p\|_\infty \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T}). \tag{0.4}$$

2. Choose $p_n \in \mathscr{P}_n(\mathbb{T})$ such that $\|f - p_n\|_\infty \leqslant 2Cn^{-\alpha}$ for $n \in \mathbb{N}$. Define $q_0 = p_1$, and $q_n = p_{2^n} - p_{2^{n-1}}$ for $n \in \mathbb{N}$.
   (a) Show that $\sum\limits_{n=0}^{\infty} q_n = f$ and the convergence is uniform.
   (b) Show that
$$\big|q_n(x) - q_n(y)\big| \leqslant 6Cn2^{n(1-\alpha)}|x - y| \quad \text{and} \quad \big|q_n(x) - q_n(y)\big| \leqslant 12C2^{-n\alpha}.$$

(c) For any $x, y \in \mathbb{T}$ with $|x-y| \leqslant 1$, choose $m \in \mathbb{N}$ such that $2^{-m} \leqslant |x-y| \leqslant 2^{1-m}$. Then use the inequality

$$\big|f(x)-f(y)\big| \leqslant \sum_{n=0}^{m-1} \big|q_n(x)-q_n(y)\big| + \sum_{n=m}^{\infty} \big|q_n(x)-q_n(y)\big|$$

to show that $\big|f(x)-f(y)\big| \leqslant B|x-y|^{\alpha}$ for some constant $B>0$.

**Hint of 1**: Suppose the contrary that there exists $p \in \mathscr{P}_n(\mathbb{T})$ such that $\|p'\|_\infty > n\|p\|_\infty$. By rescaling $p$ and relabeling points in $\mathbb{T}$ if necessary, without loss of generality we can assume that

$$\|p'\|_\infty > n\,, \quad \|p\|_\infty < 1\,, \quad \text{and} \quad p'(0) = \|p'\|_\infty\,.$$

Choose $\gamma \in \big[-\frac{\pi}{n}, \frac{\pi}{n}\big]$ such that $\sin(n\gamma) = -p(0)$ and $\cos(n\gamma) > 0$, and define $r(x) = \sin n(x-\gamma) - p(x)$. Show that $r$ has at least $2n+2$ distinct zeros in $\big(-\pi+\gamma+\frac{\pi}{2n}, \pi+\gamma+\frac{\pi}{2n}\big)$ by showing that $r$ has at least one zero in $(\alpha_k, \alpha_{k+1})$, where $\alpha_k = \gamma + \frac{\pi}{n}\big(k+\frac{1}{2}\big)$ for each $|k| \leqslant n$, while $r$ has at least 3 distinct zeros in $(\alpha_s, \alpha_{s+1})$ if $0 \in (\alpha_s, \alpha_{s+1})$. On the other hand, the fact that $r \in \mathscr{P}_n(\mathbb{T})$ implies that $r$ has at most $2n$ distinct zeros in $\mathbb{T}$ unless $r$ is the zero function which leads to a contradiction.

**Problem 7.** 1. Let $\{a_k\}_{k=1}^\infty$ be a sequence, and $\{b_n\}_{n=1}^\infty$ be the Cesàro mean of $\{a_k\}_{k=1}^\infty$; that is, $b_n = \frac{1}{n}\sum\limits_{k=1}^{n} a_k$. Show that if $\{a_k\}_{k=1}^\infty$ converges to $a$, then $\{b_n\}_{n=1}^\infty$ converges to $a$.

2. Let $\{f_k\}_{k=1}^\infty$ be a sequence of bounded real-valued functions defined on $A$, and $\{g_n\}_{n=1}^\infty$ be the Cesàro mean of $\{f_k\}_{k=1}^\infty$; that is, $g_n = \frac{1}{n}\sum\limits_{k=1}^{n} f_k$. Show that if $\{f_k\}_{k=1}^\infty$ converges uniformly to $f$ on $B \subseteq A$ and $f$ is bounded on $B$, then $\{g_n\}_{n=1}^\infty$ converges uniformly to $f$ on $B$.

*Proof.* 1. Let $\varepsilon > 0$ be given. Since $\lim\limits_{k\to\infty} a_k = a$, there exists $N_1 > 0$ such that

$$|a_k - a| < \frac{\varepsilon}{2} \quad \text{whenever} \quad k \geqslant N_1\,.$$

Since $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{N_1} |a_k - a| = 0$, there exists $N_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{N_1} |a_k - a| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N_2\,.$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$,

$$\begin{aligned} |b_n - a| &= \Big|\frac{1}{n}\sum_{k=1}^{n} a_k - a\Big| \leqslant \frac{1}{n}\sum_{k=1}^{n} |a_k - a| \leqslant \frac{1}{n}\sum_{k=1}^{N_1} |a_k - a| + \frac{1}{n}\sum_{k=N_1}^{n} |a_k - a| \\ &< \frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=N_1}^{n} \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{n-N_1+1}{n} < \varepsilon\,. \end{aligned}$$

2. Suppose that $\left|f_k(x)\right| \leqslant M_k$ and $\left|f(x)\right| \leqslant M$ for all $x \in B$. Since $\{f_k\}_{k=1}^\infty$ converges uniformly to $f$ on $B$, there exists $N_1 > 0$ such that

$$\left|f_k(x) - f(x)\right| < \frac{\varepsilon}{2} \qquad \forall\, k \geqslant N_1 \text{ and } x \in B\,.$$

If $x \in B$, by the fact that

$$\sum_{k=1}^{N_1} \left|f_k(x) - f(x)\right| \leqslant \sum_{k=1}^{N_1} (M_k + M) < \infty\,,$$

we find that $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{N_1} \|f_k - f\|_\infty = 0$; thus there exists $N_2 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{N_1} \left|f_k(x) - f(x)\right| < \frac{\varepsilon}{2} \quad \text{whenever} \quad n \geqslant N_2 \text{ and } x \in B\,.$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geqslant N$ and $x \in B$,

$$\begin{aligned}\left|g_n(x) - f(x)\right| &= \left|\frac{1}{n}\sum_{k=1}^{n} f_k(x) - f(x)\right| \leqslant \frac{1}{n}\sum_{k=1}^{N_1} \left|f_k(x) - f(x)\right| + \frac{1}{n}\sum_{k=N_1}^{n} \left|f_k(x) - f(x)\right| \\ &< \frac{\varepsilon}{2} + \frac{1}{n}\sum_{k=N_1}^{n} \frac{\varepsilon}{2} < \varepsilon\,;\end{aligned}$$

thus $\{g_n\}_{n=1}^\infty$ converges uniformly to $f$ on $B$. □

**Problem 8.** Let $f \in \mathscr{C}(\mathbb{T})$ and $\{\widehat{f}_k\}_{k=-\infty}^\infty$ be the Fourier coefficients defined in Remark 8.2 in the lecture note. Show that if $\sum\limits_{k=-\infty}^\infty |\widehat{f}_k| < \infty$, then the Fourier series of $f$ converges uniformly to $f$ on $\mathbb{R}$.

*Proof.* Let $M_k = |\widehat{f}_k|$ and $\sum\limits_{k=-\infty}^\infty |\widehat{f}_k| = M$. Then $\left|s_n(f,x)\right| \leqslant M$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Moreover,

$$\left|\widehat{f}_k e^{ikx}\right| \leqslant M_k \quad \forall\, x \in \mathbb{R} \qquad \text{and} \qquad \sum_{k=-\infty}^\infty M_k = M < \infty\,.$$

Therefore, the Weierstrass $M$-test implies that the Fourier series converges uniformly on $\mathbb{R}$. Suppose that the Fourier series converges uniformly to $g$. Then $\left|g(x)\right| \leqslant M$ for all $x \in \mathbb{R}$; thus Problem 7 implies that the Cesàro mean of $\{s_k(f,\cdot)\}_{k=1}^\infty$ converges uniformly to $g$ on $\mathbb{R}$. Since $f \in \mathscr{C}(\mathbb{T})$, the Cesàro mean of the Fourier series of $f$ converges uniformly to $f$ on $\mathbb{R}$; thus $f = g$. □