Exercise Problem Sets 5

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Problem 1. Determine which of the following real series $\sum_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

1. $g_k(x) = \begin{cases} 0 & \text{if } x \leqslant k\,, \\ (-1)^k & \text{if } x > k\,. \end{cases}$

2. $g_k(x) = \begin{cases} \dfrac{1}{k^2} & \text{if } |x| \leqslant k\,, \\ \dfrac{1}{x^2} & \text{if } |x| > k\,. \end{cases}$

3. $g_k(x) = \dfrac{(-1)^k}{\sqrt{k}} \cos(kx)$ on $\mathbb{R}$.

Proof. 1. By the definition of g_k, we find that the partial sum $S_n(x) = \sum_{k=1}^{n} g_k(x)$ satisfies that for all $n \in \mathbb{N}$,

$$S_{2n}(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-1,2n]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$S_{2n-1}(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-3,2n-2] \cup (2n-1,\infty)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ converges pointwise to the function

$$S(x) = \begin{cases} -1 & \text{if } x \in (1,2] \cup (3,4] \cup \cdots \cup (2n-3,2n-2] \cup \cdot\,, \\ 0 & \text{otherwise} \end{cases}$$

or more precisely,

$$S(x) = \sum_{k=1}^{\infty} \mathbf{1}_{(2k-1,2k]}(x)\,.$$

The convergence is uniformly on any bounded subset of $\mathbb{R}$, and the limit function S has discontinuities on $\mathbb{N}$.

2. Let $M_k = \dfrac{1}{k^2}$. Then $\sup_{x\in\mathbb{R}} |g_k(x)| \leqslant M_k$ and $\sum_{k=1}^{\infty} M_k$ converges (by the integral test). Therefore, the Weierstrass M-test implies that $\sum_{k=1}^{\infty} g_k$ converges uniformly on $\mathbb{R}$.

3. If $x = (2n+1)\pi$ for some $n \in \mathbb{Z}$, then $\cos(kx) = (-1)^k$ for all $k \in \mathbb{N}$; thus $\sum_{k=1}^{\infty} g_k(x)$ diverges at $x = (2n+1)\pi$ (by the integral test).

Now suppose that $x \notin \{(2n+1)\pi \mid n \in \mathbb{Z}\}$. Let $S_n(x) = \sum\limits_{k=1}^{n} (-1)^k \cos(kx)$. Then $S_n(x) = \sum\limits_{k=1}^{n} \cos(k(x+\pi))$ and

$$\begin{aligned} 2\sin\frac{x+\pi}{2} S_n(x) &= \sum_{k=1}^{n} \Big[\sin\big(k+\frac{1}{2}\big)(x+\pi) - \sin\big(k-\frac{1}{2}\big)(x+\pi)\Big] \\ &= \sin\big(n+\frac{1}{2}\big)(x+\pi) - \sin\frac{x+\pi}{2}\,; \end{aligned}$$

thus

$$S_n(x) = \frac{(-1)^n \cos(n+\frac{1}{2})x}{2\cos\frac{x}{2}} - \frac{1}{2} \qquad \forall\, x \in \mathbb{R}\backslash\{(2n+1)\pi \mid n \in \mathbb{Z}\}\,.$$

The equality above shows that

$$|S_n(x)| \leqslant \frac{1}{2|\cos\frac{x}{2}|} + \frac{1}{2} \qquad \forall\, x \in \mathbb{R}\backslash\{(2n+1)\pi \mid n \in \mathbb{Z}\}\,,$$

which is bounded independent of n. The Dirichlet test then shows that $\sum\limits_{k=1}^{\infty} g_k(x)$ converges for all $x \in \mathbb{R}\backslash\{(2n+1)\pi \mid n \in \mathbb{Z}\}$. Therefore, $\sum\limits_{k=1}^{\infty} g_k$ converges pointwise on $\mathbb{R}\backslash\{(2n+1)\pi \mid n \in \mathbb{Z}\}$.

Let $A \subseteq \mathbb{R}$ be a set satisfying that

$$d\big(x, \{(2n+1)\pi \mid n \in \mathbb{Z}\}\big) = \inf\big\{|x-y| \,\big|\, y \in \{(2n+1)\pi \mid n \in \mathbb{Z}\}\big\} \geqslant \delta \qquad \forall\, x \in A\,.$$

Then the computation above shows that $|S_n(x)| \leqslant R \equiv \dfrac{1}{2|\cos\frac{\delta}{2}|} + \dfrac{1}{2}$ for all $x \in A$. If $n > m$, we have

$$\begin{aligned} \sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx) &= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} \big[S_k(x) - S_{k-1}(x)\big] \\ &= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_k(x) - \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_{k-1}(x) \\ &= \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}} S_k(x) - \sum_{k=m}^{n-1} \frac{1}{\sqrt{k+1}} S_k(x) \\ &= \frac{1}{\sqrt{n}} S_n(x) - \frac{1}{\sqrt{m+1}} S_m(x) + \sum_{k=m+1}^{n-1} \Big(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\Big) S_k(x)\,; \end{aligned}$$

thus if $x \in A$,

$$\Big|\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx)\Big| \leqslant \Big[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m+1}} + \sum_{k=m+1}^{n-1} \Big(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\Big)\Big] R = \frac{2R}{\sqrt{m+1}}\,.$$

Therefore, for a given $\varepsilon > 0$, by choosing $N > 0$ satisfying $\dfrac{2R}{\sqrt{N+1}} < \varepsilon$ we conclude that

$$\Big|\sum_{k=m+1}^{n} \frac{(-1)^k}{\sqrt{k}} \cos(kx)\Big| < \varepsilon \quad \text{whenever } n > m \geqslant N \text{ and } x \in A\,.$$

By the Cauchy criterion, $\sum\limits_{k=1}^{\infty} g_k$ converges uniformly on A; thus $\sum\limits_{k=1}^{\infty} g_k$ is continuous at every point at which the series converges. □

Problem 2. Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{R}$ be a real sequence, and $f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ be a power series with radius of convergence $R > 0$. Let $s_n(x) = \sum\limits_{k=0}^{n} a_k x^k$ be the n-th partial sum, $R_n(x) = f(x) - s_n(x)$, and $g(x) = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$. For $x, x_0 \in [-\rho, \rho] \subsetneq (-R, R)$, where $x \neq x_0$, write

$$\frac{f(x) - f(x_0)}{x - x_0} - g(x_0) = \frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0) + \big(s_n'(x_0) - g(x_0)\big) + \frac{R_n(x) - R_n(x_0)}{x - x_0}. \qquad (\star)$$

1. Show that
$$\Big|\frac{R_n(x) - R_n(x_0)}{x - x_0}\Big| \leqslant \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1},$$
and use the inequality above to show that $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = g(x_0)$.

2. Generalize the conclusion to complex power series: suppose that $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$ and the power series $\sum\limits_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$; that is, $\sum\limits_{k=0}^{\infty} a_k z^k$ converges for all $|z| < R$ but for each $n \in \mathbb{N}$ there exists z_n with $|z_n - c| > R + \dfrac{1}{n}$ such that $\sum\limits_{k=0}^{\infty} a_k z_n^k$ diverges. Show that
$$\frac{d}{dz} \sum_{k=0}^{\infty} a_k z^k = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad \forall\, |z| < R.$$
Assume that you have known $\dfrac{d}{dz} \sum\limits_{k=0}^{n} a_k z^k = \sum\limits_{k=1}^{n} k a_k z^{k-1}$ for all $n \in \mathbb{N} \cup \{0\}$ (this can be proved using the definition of differentiability of functions with values in normed vector spaces provided in Chapter 5).

Proof. Let R be the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k x^k$.

Claim: The series $\sum\limits_{k=1}^{\infty} k|a_k|\rho^{k-1}$ converges for all $0 < \rho < R$.

To see the claim, we note that for each $0 < r < R$, $\sum\limits_{k=0}^{\infty} a_k r^k$ converges; thus $\lim\limits_{k \to \infty} a_k r^k = 0$. This implies that the sequence $\{a_k r^k\}_{k=1}^{\infty}$ is bounded for all $0 < r < R$. Let $M(r)$ denote a real number satisfying $|a_k r^k| \leqslant M(r)$ for all $k \in \mathbb{N} \cup \{0\}$. Then for $0 < \rho < R$, we choose r so that $0 < \rho < r < R$ so that

$$\sum_{k=1}^{\infty} k|a_k|\rho^{k-1} = \sum_{k=1}^{\infty} k|a_k| r^{k-1} \Big(\frac{\rho}{r}\Big)^{k-1} \leqslant M(r) \sum_{k=1}^{\infty} k \Big(\frac{\rho}{r}\Big)^{k-1}$$

where the convergence of the series on the right-hand side can be obtained by the ratio test ((5) of Theorem 2.70). The claim is then established by the comparison test ((2) of Theorem 2.70).

1. Since $R_n(x) = \sum\limits_{k=n+1}^{\infty} a_k x^k$ converges for all $x \in (-R, R)$, for $x \neq x_0$ we have

$$\frac{R_n(x) - R_n(x_0)}{x - x_0} = \frac{1}{x - x_0} \sum_{k=n+1}^{\infty} a_k (x^k - x_0^k) = \sum_{k=n+1}^{\infty} a_k \big(x^{k-1} + x^{k-2} x_0 + \cdots + x x_0^{k-2} + x_0^{k-1}\big);$$

thus if $x, x_0 \in [-\rho, \rho] \subseteq (-R, R)$ and $x \neq x_0$,

$$\left|\frac{R_n(x) - R_n(x_0)}{x - x_0}\right| \leqslant \sum_{k=n+1}^{\infty} |a_k|\big(|x|^{k-1} + |x|^{k-2}|x_0| + \cdots + |x||x_0|^{k-2} + |x_0|^{k-1}\big)$$
$$\leqslant \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}\,.$$

Let $\varepsilon > 0$ be given. By the claim above there exists $N > 0$ such that

$$\sum_{k=n+1}^{\infty} k|a_k||x_0|^{k-1} < \frac{\varepsilon}{3} \qquad \text{and} \qquad \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1} < \frac{\varepsilon}{3}\,.$$

Therefore, $(\star)$ implies that

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - g(x_0)\right| \leqslant \left|\frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0)\right| + \left|s_n'(x_0) - g(x_0)\right| + \left|\frac{R_n(x) - R_n(x_0)}{x - x_0}\right|$$
$$\leqslant \left|\frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0)\right| + \left|\sum_{k=n+1}^{\infty} k a_k x_0^{k-1}\right| + \sum_{k=n+1}^{\infty} k|a_k|\rho^{k-1}$$
$$\leqslant \left|\frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0)\right| + \frac{2\varepsilon}{3}\,;$$

thus

$$\limsup_{x\to x_0} \left|\frac{f(x) - f(x_0)}{x - x_0} - g(x_0)\right| \leqslant \limsup_{x\to x_0} \left|\frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0)\right| + \frac{2\varepsilon}{3}$$
$$= \lim_{x\to x_0} \left|\frac{s_n(x) - s_n(x_0)}{x - x_0} - s_n'(x_0)\right| + \frac{2\varepsilon}{3} < \varepsilon\,.$$

Since $\varepsilon > 0$ is given arbitrarily, we find that $\lim\limits_{x\to x_0} \left|\dfrac{f(x) - f(x_0)}{x - x_0} - g(x_0)\right| = 0$ which shows that

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g(x_0)\,.$$

□

Problem 3. Let $\{a_k\}_{k=0}^{\infty} \subseteq \mathbb{C}$, $c \in \mathbb{C}$, $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ be a power series with radius of convergence $R > 0$; that is, $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges for all $x \in B(c, R)$ but for each $n \in \mathbb{N}$ there exists x_n with $|x_n - c| > R + \dfrac{1}{n}$ such that $\sum\limits_{k=0}^{\infty} a_k(x_n - c)^k$ diverges. Let $K \subseteq B(c, R)$ be a compact set. Show that

1. The power series $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on K.

2. The power series $\sum\limits_{k=0}^{\infty} (k+1)a_{k+1}(x-c)^k$ converges pointwise on $B(c, R)$, and converges uniformly on K.

Proof. 1. Since $K \subseteq B(c, R)$ is compact, there exists $r > 0$ such that $K \subseteq B[c, \rho] \subseteq B(c, R)$. In fact, $\rho = \sup\limits_{x\in K} |x - c|$ will do the job. It then suffices to show that $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on $B[c,\rho]$. Let $r = \dfrac{\rho+R}{2}$. Then $c+r \in B(c,R)$ so that the series $\sum\limits_{k=0}^{\infty} a_k r^k$ converges; thus $\lim\limits_{k\to\infty} a_k r^k = 0$. Therefore, there exists $M(r) > 0$ such that

$$|a_k| r^k \leqslant M(r) \qquad \forall\, k \in \mathbb{N}\,.$$

Since

$$\sum_{k=0}^{\infty} |a_k|\rho^k = \sum_{k=0}^{\infty} |a_k| r^k \Big(\frac{\rho}{r}\Big)^k \leqslant M(r) \sum_{k=0}^{\infty} \Big(\frac{\rho}{r}\Big)^k$$

and the series on the right-hand side converges because of the geometric series test ((1) of Theorem 2.70), the comparison test shows that $\sum\limits_{k=0}^{\infty} |a_k|\rho^k$ converges. Therefore, for each $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that

$$\sum_{k=n+1}^{\infty} |a_k|\rho^k < \varepsilon \qquad \forall\, n \geqslant N(\varepsilon)\,.$$

As a consequence, for a given $\varepsilon > 0$, if $x \in B[c,\rho]$ and $n > m \geqslant N(\varepsilon)$,

$$\Big|\sum_{k=m+1}^{n} a_k (x-c)^k\Big| \leqslant \sum_{k=m+1}^{n} |a_k|\rho^k < \varepsilon$$

which, by the Cauchy criteria, shows that the power series $\sum\limits_{k=0}^{\infty} a_k (x-c)^k$ converges uniformly on $B[c,\rho]$.

2. The proof of the pointwise convergence on $B(c,R)$ is exactly the same as the claim in Problem 2, and the proof of the uniform convergence on K is the same as the proof in part 1, and we omit here. □

Problem 4. Suppose that the series $\sum\limits_{n=0}^{\infty} a_n = 0$, and $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ for $-1 < x \leqslant 1$. Show that f is continuous at $x = 1$ by complete the following.

1. Write $s_n = a_0 + a_1 + \cdots + a_n$ and $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$. Show that

$$S_n(x) = (1-x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n$$

and $f(x) = (1-x) \sum\limits_{n=0}^{\infty} s_n x^n$.

2. Using the representation of f from above to conclude that $\lim\limits_{x\to 1^-} f(x) = 0$.

3. What if $\sum\limits_{n=0}^{\infty} a_n$ is convergent but not zero?

Proof. 1. Let $s_n = a_0 + a_1 + \cdots + a_n$ and $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

$$\begin{aligned}
S_n(x) &= \sum_{k=0}^{n} a_k x^k = a_0 + \sum_{k=1}^{n} a_k x^k = s_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k \\
&= s_0 + \sum_{k=1}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k = \sum_{k=0}^{n} s_k x^k - \sum_{k=0}^{n-1} s_k x^{k+1} \\
&= s_n x^n + \sum_{k=0}^{n-1} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k \\
&= (1-x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n \,.
\end{aligned}$$

Therefore, by the fact that $\lim\limits_{n\to\infty} s_n = 0$, we find that if $x \in (-1, 1]$,

$$f(x) = \lim_{n\to\infty} S_n(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k \,.$$

2. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to\infty} s_n = 0$, there exists $N > 0$ such that $|s_n| < \dfrac{\varepsilon}{2}$ for all $n \geqslant N$. Choose $0 < \delta < 1$ such that $\delta \sum\limits_{k=0}^{N-1} |s_k| < \dfrac{\varepsilon}{2}$. Then if $1 - \delta < x < 1$,

$$\begin{aligned}
\big|f(x)\big| &\leqslant |1-x| \sum_{k=0}^{N-1} |s_k||x|^k + |1-x| \sum_{k=N}^{\infty} |s_k||x|^k \\
&\leqslant \delta \sum_{k=0}^{N-1} |s_k| + \frac{\varepsilon}{2} |1-x||x|^N \sum_{k=0}^{\infty} |x|^k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} |1-x| \frac{1}{1-|x|} = \varepsilon \,.
\end{aligned}$$

Therefore, $\lim\limits_{x\to 1^-} f(x) = 0 = f(1)$ which shows that f is continuous at 1.

3. If $s = \sum\limits_{k=0}^{\infty} a_k \neq 0$, we define a new series $\sum\limits_{n=0}^{\infty} b_n x^n$ by $b_0 = a_0 - s$ and $b_n = a_n$ for all $n \in \mathbb{N}$. Then $g(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ also converges for $x \in (-1, 1]$ and satisfies that $g(1) = 0$. Therefore, 1 and 2 imply that g is continuous at 1; thus $\lim\limits_{x\to 1^-} g(x) = 0$. By the fact that $g(x) = f(x) - s$, we conclude that

$$\lim_{x\to 1^-} f(x) = s = \sum_{n=0}^{\infty} a_n = f(1) \,.$$

□

Problem 5. Let $\delta : (\mathscr{C}([-1, 1]; \mathbb{R}), \|\cdot\|_\infty) \to \mathbb{R}$ be defined by $\delta(f) = f(0)$. Show that δ is linear and uniformly continuous.

Proof. Let $c \in \mathbb{R}$ and $f, g \in \mathscr{C}([-1, 1]; \mathbb{R})$. Then

$$\delta(cf + g) = cf(0) + g(0) = c\delta(f) + \delta(g)$$

which shows that δ is linear on $\mathscr{C}([-1, 1]; \mathbb{R})$.

For the uniform continuity of δ, let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then if $\|f - g\|_\infty < \delta$, we have

$$\big|f(0) - g(0)\big| \leqslant \|f - g\|_\infty < \delta = \varepsilon$$

which implies that δ is uniformly continuous. □

Problem 6. Let (M, d) be a metric space, and $K \subseteq M$ be a compact subset.

1. Show that the set $U = \{f \in \mathscr{C}(K;\mathbb{R}) \,|\, a < f(x) < b \text{ for all } x \in K\}$ is open in $(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty)$ for all $a, b \in \mathbb{R}$.

2. Show that the set $F = \{f \in \mathscr{C}(K;\mathbb{R}) \,|\, a \leqslant f(x) \leqslant b \text{ for all } x \in K\}$ is closed in $(\mathscr{C}(K;\mathbb{R}), \|\cdot\|_\infty)$ for all $a, b \in \mathbb{R}$.

3. Let $A \subseteq M$ be a subset, not necessarily compact. Prove or disprove that the set $B = \{f \in \mathscr{C}_b(A;\mathbb{R}) \,|\, f(x) > 0 \text{ for all } x \in A\}$ is open in $(\mathscr{C}_b(A;\mathbb{R}), \|\cdot\|_\infty)$.

Proof. 1. Let $g \in U$. By the Extreme Value Theorem, there exists $x_0, x_1 \in K$ such that

$$g(x_0) = \inf_{x\in K} g(x) \qquad \text{and} \qquad g(x_1) = \sup_{x\in K} g(x)\,.$$

Therefore, $a < \inf\limits_{x\in K} g(x) \leqslant \sup\limits_{x\in K} g(x) < b$. Let $r = \min\{b - \sup\limits_{x\in K} g(x), \inf\limits_{x\in K} g(x) - a\}$. Then $r > 0$. Moreover, if $f \in B(g, r)$ and $x \in K$, we have

$$|f(x) - g(x)| \leqslant \sup_{x\in K} |f(x) - g(x)| = \|f - g\|_\infty < r\,.$$

Therefore, if $f \in B(g, r)$, by the fact that $r \leqslant b - \sup\limits_{x\in K} g(x)$ and $r \leqslant \inf\limits_{x\in K} g(x) - a$, we conclude that if $x \in K$,

$$a \leqslant \inf_{x\in K} g(x) - r \leqslant g(x) - r < f(x) < g(x) + r \leqslant \sup_{x\in K} g(x) + r \leqslant b$$

which implies that $f \in U$. Therefore, $B(g, r) \subseteq U$; thus U is open.

2. Let $\{f_n\}_{n=1}^\infty$ be a sequence in F such that $\{f_n\}_{n=1}^\infty$ converges uniformly to f on K. Then $f \in \mathscr{C}(K;\mathbb{R})$. Moreover, by the fact that $a \leqslant f_n(x) \leqslant b$ for all $x \in K$ and $n \in \mathbb{N}$, we find that $a \leqslant f(x) \leqslant b$ for all $x \in K$ since $f(x) = \lim\limits_{n\to\infty} f_n(x)$. This implies that $f \in F$; thus F is closed (since it contains all the limit points).

3. Consider the case $A = (0, 1)$. Then the function $f(x) = x$ belongs to B; however, for every $r > 0$, the function $g(x) = f(x) - \dfrac{r}{2}$ belongs to $B(f, r)$ since

$$\|f - g\|_\infty = \sup_{x\in(0,1)} |f(x) - g(x)| = \frac{r}{2} < r\,.$$

However, $g \notin B$ since if $0 < x \ll 1$, we have $g(x) < 0$. In other words, there exists no $r > 0$ such that $B(f, r) \subseteq B$; thus B is not open. □